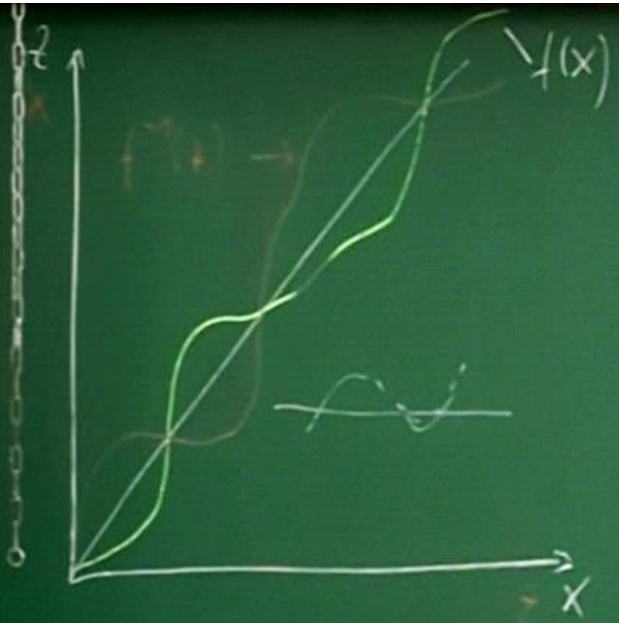
1.1 Warm up: 1-D generative models

- basic principle: $p(x)$ is complicated $\Rightarrow$ reduce it to something simpler
  new variable ("code"): $z = f(x)$, $q(z)$ is easy
  ↑ deterministic function

- for generative modeling, $f(x)$ must be invertible, $f^{-1}(z)$ exists & is unique
  - if $x \sim p(x) \Rightarrow z = f(x) \sim q(z)$ inference direction
  - if $z \sim q(z) \Rightarrow x = f^{-1}(z) \sim p(x)$ generative direction

- consequence in 1-D: $f(x)$ must be a monotonic function (convention: increasing)
  $\frac{d}{dx} f(x) \geq 0$ for all x (strictly monot: > 0)

universal property of monotonic functions
$$\left.\frac{d}{dx} f(x)\right|_{x=x_0} = \frac{1}{\left.\frac{d}{dz} f^{-1}(z)\right|_{z=f(x_0)}}$$



Proof

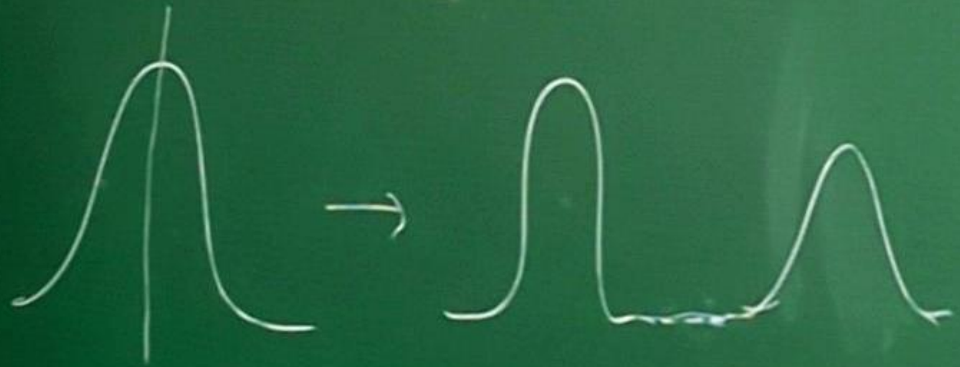
$$\frac{d}{dz} f^{-1}(z)\Big|_{z=z_0} = \lim_{z\to z_0} \frac{f^{-1}(z)-f^{-1}(z_0)}{z-z_0}$$

$$x = f^{-1}(z)$$

$$x_0 = f^{-1}(z_0)$$

$$= \lim_{x\to x_0} \frac{f^{-1}(z=f(x)) - f^{-1}(z=f(x_0))}{f(x)-f(x_0)}$$

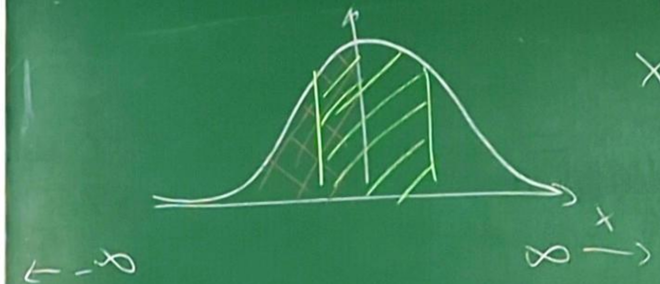
$$= \lim_{x\to x_0} \frac{x - x_0}{f(x)-f(x_0)}$$



$$= \left(\frac{d}{dx} f(x)\Big|_{x=f(z_0)}\right)^{-1}$$

- apply to probability distr.:

define mass in interval (x_0, x_1) $\quad m(x_0, x_1) = \int_{x_0}^{x_1} p(x)\,dx$



$x_0 = -\infty,\ x_1 = \infty \qquad m = 1$

$x_0 = -\infty,\ x_1 = 0 \qquad m = \frac{1}{2}$

$x_0 = -\sigma,\ x_1 = \sigma \qquad m = 0.68$

$x_0 = -2\sigma,\ x_1 = 2\sigma \qquad m = 0.95$

$f(x)$ is permissible, if for every interval (x_0, x_1): $\quad \int_{x_0}^{x_1} p(x)\,dx = \int_{z_0 = f(x_0)}^{z_1 = f(x_1)} q(z)\,dz$

substitution: $z = f(x)$, $dz = f'(x)\,dx$

$x_0 = f^{-1}(z_0) \qquad x_1 = f^{-1}(z_1)$

$$\int_{x_0}^{x_1} p(x)\,dx = \int_{f^{-1}(f(x_0))}^{f^{-1}(f(x_1))} q(z = f(x))\, f'(x)\,dx$$

must hold for any (x_0, x_1)

$$\boxed{p(x) = q(z = f(x)) \cdot f'(x)}$$

change-of-variables formula