contd: <u>Generative modelling in 1-D</u>

general idea:

1. pick a simple distribution $q(z)$ — invertible $x = f^{-1}(z)$
2. calculate (or learn) a transformation $z = f(x)$

such that
- inference via change-of-variables formula

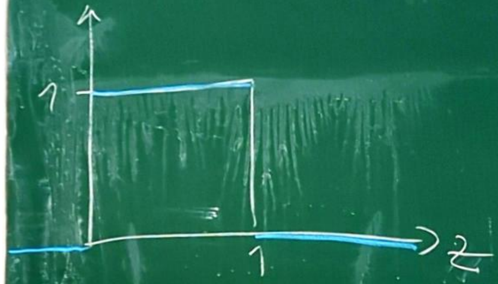
$$p(x) = q(z = f(x)) \cdot f'(x)$$

- generate via "inverse transform sampling"

$$z \sim q(z), \quad x = f^{-1}(z) \sim p(x)$$

$q(z) \sim \text{uniform}(0,1)$

all softwares contain a random generator for uniform, e.g. Mersenne twister



$\forall z \in [0,1]: \; q(z) \equiv 1$ $\Rightarrow$ CoV becomes

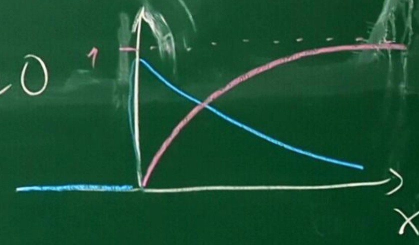
$$\boxed{p(x) = f'(x)}$$

$$\Rightarrow f(x) = \int_{-\infty}^{x} p(x)\,dx$$ - cumulative distribution function (CDF)

[$p(x)$: probability density function (PDF)]

$$z \in [0,1] \begin{cases} \text{if } x \leq x_{min} \text{ (left boundary of } p(x)\text{)} & f(x) = 0 \\ \text{if } x \geq x_{max} \text{ (right —||—)} & f(x) = 1 \quad \text{(normalization of } p(x)\text{)} \\ \text{if } x_{min} < x < x_{max} & 0 < f(x) < 1 \quad \text{because } p(x) \geq 0 \end{cases}$$

<u>ex:</u> exponential distribution $p(x) = \begin{cases} \frac{1}{?} e^{-x/\tau} & \text{if } x \geq 0 \\ 0 & \text{else} \end{cases}$ ← normalization



$$1 = \int_0^\infty e^{-x'/\tau}\,dx' = -\tau e^{-x'/\tau}\Big|_0^\infty$$

$$= -\tau\left(e^{-\infty/\tau} - e^{-0/\tau}\right) = \tau \Rightarrow p(x) = \frac{1}{\tau} e^{-x/\tau} \quad \text{for } x \geq 0$$

$$z = f(x) = \int_0^x \frac{1}{\tau} e^{-x'/\tau}\,dx' = -\left(e^{-x/\tau} - 1\right) = 1 - e^{-x/\tau}$$ CDF of exponential

$$\boxed{f^{-1}(z) = -\tau \log(1-z)}$$ inverse CDF or quantile fct. or percent point fct. (PPF)

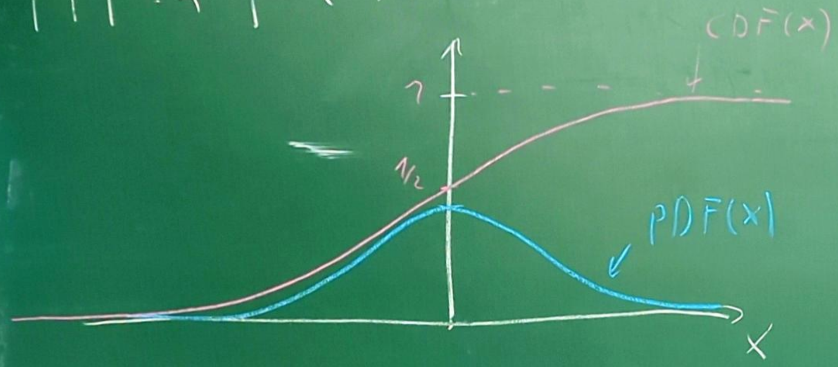
ex: Gaussian distribution

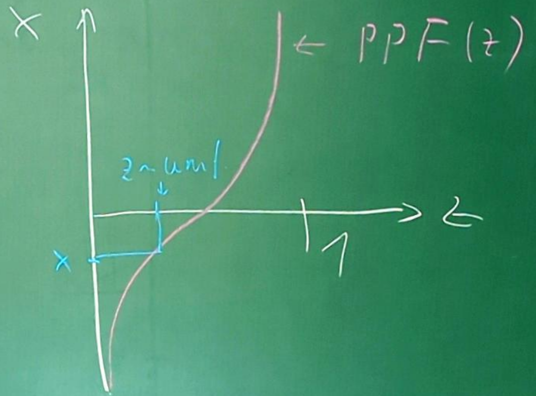
$$\text{PDF: } p(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{x^2}{2\sigma^2}} = \mathcal{N}(\overset{\text{mean}}{0}, \overset{\text{variance}}{\sigma^2})$$

$$\text{CDF: } z = f(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{x'^2}{2\sigma^2}}\, dx' = \frac{1}{2} + \frac{1}{2}\, \overset{\text{error function}}{\mathrm{erf}}\left(\frac{x}{\sqrt{2}\,\sigma}\right)$$

$$\text{PPF: } x = f^{-1}(z) = \sqrt{2}\,\sigma\, \mathrm{erfinv}(2z-1)$$

$$z \sim \mathrm{uniform}(0,1) \Rightarrow x \sim \mathcal{N}(0, \sigma^2)$$





$f(x)$ and $f^{-1}(z)$ are not required to be continuous (= may have jumps)

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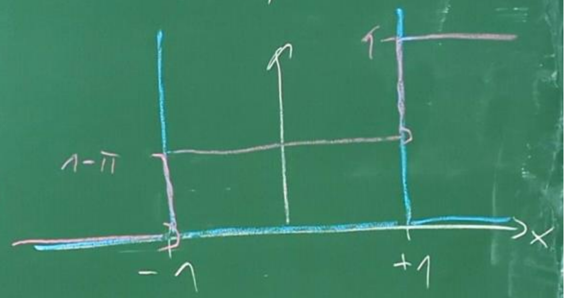
ex: Bernoulli distribution $\quad x \sim p(x) \Leftrightarrow \begin{cases} +1 & \text{with probability } \pi \in [0,1] \\ -1 & \text{— '' —} \quad 1-\pi \end{cases}$

continuous embedding via the δ-distribution

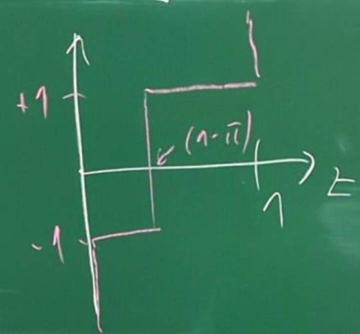
$$p(x) = (1-\pi)\,\delta(x+1) + \pi\,\delta(x-1)$$



for any function $h(x)$, $\quad \int_{-\infty}^{\infty} h(x')\,\delta(x-x')\,dx' = h(x)$

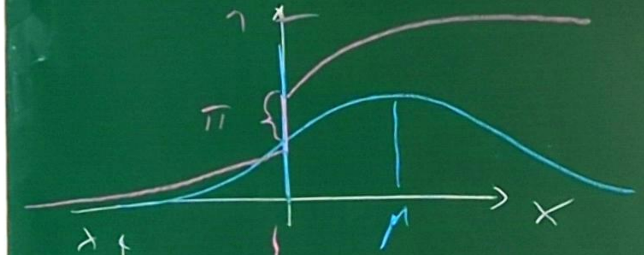
$$\text{CDF} \quad f(x) = \int_{-\infty}^{x} p(x')\,dx' = \int_{-\infty}^{x} \left[(1-\pi)\,\delta(x'+1) + \pi\,\delta(x'-1)\right] dx'$$

alg: $\quad z \sim \mathrm{Unif}([0,1])\,, \quad x = \begin{cases} -1 & \text{if } z \le 1-\pi \\ +1 & \text{else} \end{cases}$



ex: spike-and-slab distribution

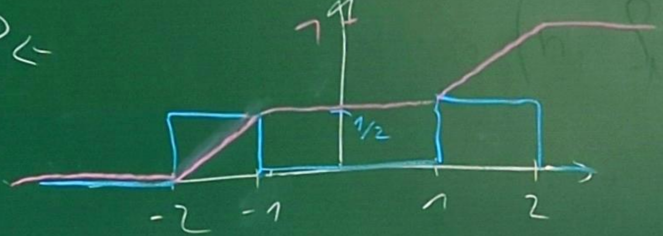
$$x \sim p(x) \iff x = \begin{cases} 0 & \text{with probability } \pi \quad (\text{"nothing happened"}) \\ \mathcal{N}(\mu, \sigma^2) & \text{with prob } (1-\pi) \quad (\text{"actual measurement"}) \end{cases}$$

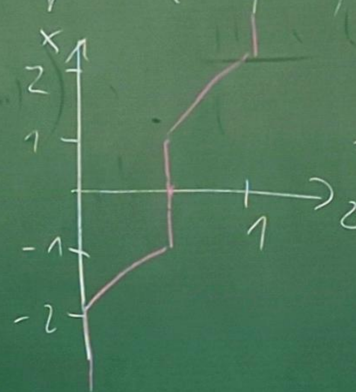


$$p(x) = \pi\, \delta(x) + (1-\pi)\, \mathcal{N}(x \mid \mu, \sigma^2)$$

ex two disconnected islands

$$p(x) = \frac{1}{2}\, \text{uniform}(-2,-1) + \frac{1}{2}\, \text{uniform}(1,2)$$





discontinuous PPF

$\Rightarrow$ define tie breaker for jumps arbitrarily

(if $z = \frac{1}{2}$, $x \to -1$