Robotics 2020/21

I lecture

What is a matrix?

A matrix $m \times n$ is a table of $m \cdot n$ elements ordered in m rows and n columns

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

- If:
  - $m \neq n \Rightarrow$ rectangular matrix
  - $m = n \Rightarrow$ squared matrix
  - $m = 1 \vee n = 1 \Rightarrow$ vector

What is the transpose of a matrix?

Given A matrix A^T is the matrix obtained swapping in order the rows and the colums of A.

Ex.

$$A = \begin{pmatrix} 2 & 3 \\ 1 & 0 \\ 5 & \pi \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} 2 & 1 & 5 \\ 3 & 0 & \pi \end{pmatrix}$$

Basic operations:

- SUM $\forall$ only if they have same dimensions $m \times n$

$$A + B = C \Rightarrow a_{ij} + b_{ij} = c_{ij}$$

- PRODUCT with a scalar

$$\lambda A = (\lambda a_{ij}) \quad \forall i, j$$

- PRODUCT between matrices

ROWS x COLUMNS

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 3 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 3 \\ 1 & 2 \\ 0 & 0 \end{pmatrix} \Rightarrow A \cdot B = \begin{pmatrix} 2 & 7 \\ 3 & 6 \end{pmatrix}$$

! $A \cdot B \neq B \cdot A$ ORDER IS IMPORTANT

$$B \cdot A = \begin{pmatrix} 0 & 9 & 6 \\ 1 & 8 & 8 \\ 0 & 0 & 0 \end{pmatrix}$$

What is an ORTHOGONAL matrix?

A is ortogonal if $A \cdot A^T = I$ with $I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$

$\Rightarrow A^T = A^{-1}$; $\det(A) = 1 \vee -1$

if A ortogonal and B ortogonal $\Rightarrow C = A \cdot B$ is also

Inverse of a matrix A' or A^{-1}

What is the determinant of a matrix?

det(A) is a number that associats important proprieties of a matrix.

Given a square matrix A $n \times n$ $\Rightarrow$

- if $n = 1 \Rightarrow \det(A) = a_{11}$
- if $n = 2 \Rightarrow \det(A) = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$
- if $n = 3 \Rightarrow \det(A) = a_{11}(a_{22} \cdot a_{33} - a_{23} \cdot a_{32})$
$$- a_{12}(a_{21} \cdot a_{33} - a_{23} \cdot a_{31})$$
$$+ a_{13}(a_{21} a_{32} - a_{22} a_{31})$$

EX.

$$A = \begin{pmatrix} 1 & 0 & 5 \\ 2 & -1 & 0 \\ 7 & -2 & 0 \end{pmatrix}$$

$$\Rightarrow \det(A) = 1(0-0) - 0(\ldots) + 5(-4+7) = 15$$

$\boxed{\text{Inverse of a matrix}}$

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

① $\det(A) \neq 0$

$\det(A) = 2 - 1 = 1$ ok!

$\Rightarrow$ we can invert

② $A^{-1} = \dfrac{1}{\det(A)} \operatorname{Adj}(A)$ with $\operatorname{Adj}(A) = \begin{pmatrix} Cof(a_{11}) & Cof(a_{12}) \\ Cof(a_{21}) & Cof(a_{22}) \end{pmatrix}^T$

$Cof(a_{ij}) = (-1)^{i+j} \cdot \det(A_{ij})$ with A_{ij} the sub-matrix without i-row and j-column

$Cof(a_{11}) = (-1)^{1+1} \det(2) = 2$

$\ldots \Rightarrow \operatorname{Adj}(A) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^T = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

$\dfrac{1}{\det(A)} = 1 \Rightarrow A^{-1} = 1 \cdot \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

Rank of a matrix

max number of rows/columns linearly independent

How can we check that?

$$A=\begin{pmatrix} 1 & -1 & 2 \\ 2 & 3 & 4 \\ 3 & 2 & 6 \end{pmatrix}$$

$$\det(A) = 1(18-8)+1(12-12)+2(4-9) = 10-10=0$$

$\det(A)=0 \Rightarrow$ check sub-matrices

$$A\begin{pmatrix} 1 & -1 & 2 \\ 2 & 3 & 4 \\ 3 & 2 & 6 \end{pmatrix} \rightarrow \det\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} = 3+2=5 \Rightarrow \neq 0$$

$$rk(A)=2$$

$\boxed{\text{Last point}}$ PRODUCT BETWEEN VECTORS

Let's consider 2 vectors: $\vec{v}=(v_1, v_2, v_3)$ and $\vec{w}=(w_1, w_2, w_3)$ and let's call the versors $(\vec{i}, \vec{j}, \vec{k})$

$$\Rightarrow \vec{v}\times\vec{w} = (v_2w_3 - v_3w_2)\vec{i} + (v_3w_1 - v_1w_3)\vec{j} + (v_1w_2 - v_2w_1)\vec{k}$$

how to remember? $\vec{v}\times\vec{w} =$

$$\det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} = \cancel{v_2w_3\vec{i} + v_3w_1\vec{j} + v_1w_2\vec{k} - w_1v_2\vec{k}}$$

$$\vec{i}(v_2w_3 - v_3w_2) - \vec{j}(v_1w_3 - v_3w_1) + \vec{k}(\_\_\_\_)$$