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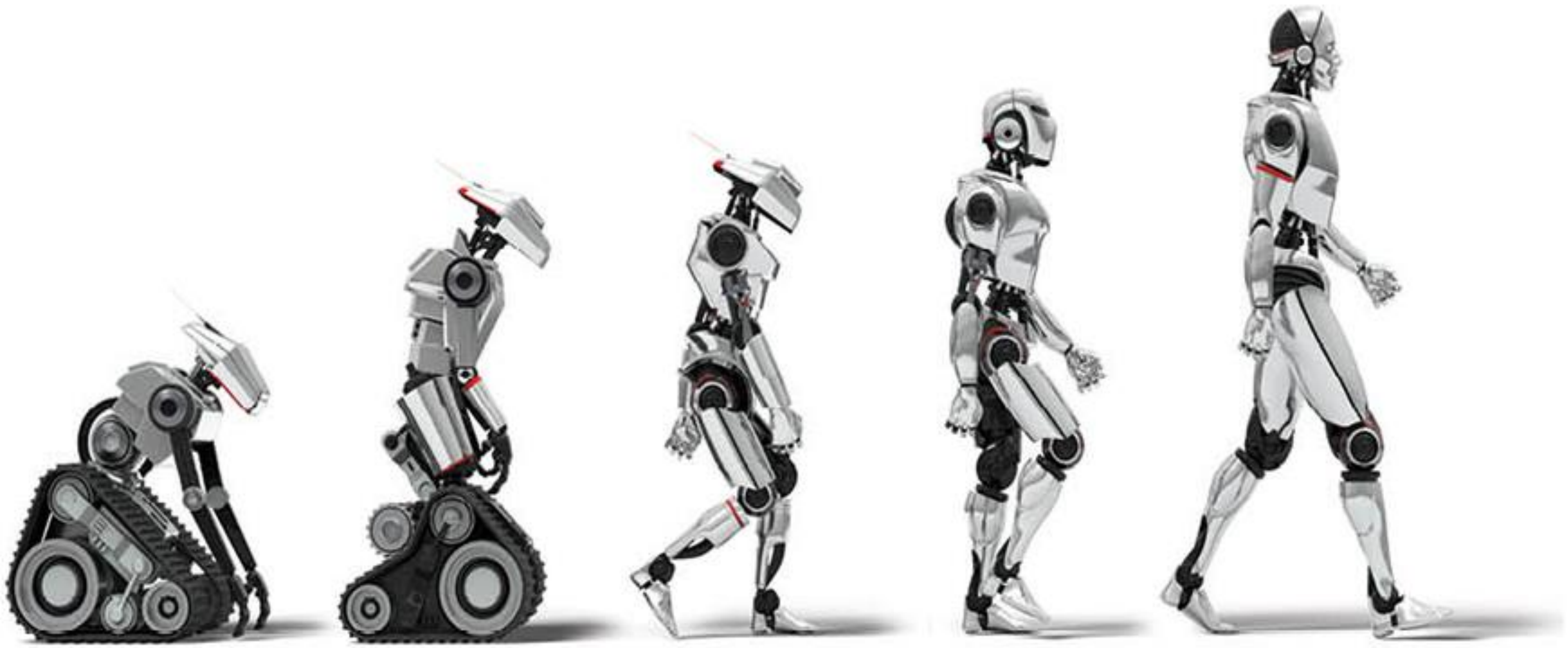
# Exercises Robotics 8



# Remarks on generation of trajectories

The trajectory with n DOF is a parameterized function q(t) with values in its motion region. The trajectory q(t) of a robot with n DOF is then a vector of n parameterized functions $q_i$(t), i ∈ {1 . . . n} with one common parameter:
q(t) = $[q_1(t), q_2(t), \ldots, q_n(t)]^T$

- A trajectory is Ck-continuous, if all derivatives up to k-th (inclusively) exist and are continuous.
- A trajectory is called smooth, if it is at least C2-continuous.
- The first derivative of trajectory related to the time ($\dot{p}$(x)) is the velocity
- The second derivative of trajectory related to the time ( $\ddot{p}$(x)) is the acceleration
- The third derivative of trajectory related to the time ($\dddot{p}$ (x)) is the jerk
- The smoothest curves are generated via infinitly often differentiable functions. e.g. e(x) , sin(x), and log(x) (for x > 0).
- Polynomials are suitable for interpolation (problem: oscillations caused by a degree that is too high).
- Piecewise polynomials with specified degree are applicable: cubic polynomial, splines, B-Splines etc.

# Trajectory planning

third-degree polynomial $\Rightarrow$ four constraints:

$$\theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

if the start and end velocity is 0 then:

$$\begin{aligned} \theta(0) &= \theta_0 \\ \theta(t_f) &= \theta_f \\ \dot{\theta}(0) &= 0 \\ \dot{\theta}(t_f) &= 0 \end{aligned}$$

# Trajectory planning

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The solution:

$$a_0 = \theta_0$$

$$a_1 = 0$$

$$a_2 = \frac{3}{t_f^2}(\theta_f - \theta_0)$$

$$a_3 = -\frac{2}{t_f^3}(\theta_f - \theta_0)$$

# Trajectory planning

similar example than above:

- positions of waypoints are given (same)
- but: velocities of waypoints are different from 0 (different)

$$
\begin{aligned}
\theta(0) &= \theta_0 \\
\theta(t_f) &= \theta_f \\
\dot{\theta}(0) &= \dot{\theta}_0 \\
\dot{\theta}(t_f) &= \dot{\theta}_f
\end{aligned}
$$

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The solution:

$$\begin{aligned}a_0 &= \theta_0\\ a_1 &= \dot{\theta}_0\\ a_2 &= \frac{3}{t_f^2}(\theta_f - \theta_0) - \frac{2}{t_f}\dot{\theta}_0 - \frac{1}{t_f}\dot{\theta}_f\\ a_3 &= -\frac{2}{t_f^3}(\theta_f - \theta_0) + \frac{1}{t_f^2}(\dot{\theta}_f + \dot{\theta}_0)\end{aligned}$$

# Trajectory planning

The 2-dof Cartesian robot in Fig. 2 should execute with its end-effector the following desired eight-shaped periodic trajectory

$$\boldsymbol{p}_d(t) = \begin{pmatrix} c + a \sin 2\omega t \\ c + b \sin \omega t \end{pmatrix}, \qquad \text{with } a, b, c, \omega > 0, \text{ for } t \in \left[0, \frac{2\pi}{\omega}\right]. \tag{1}$$

The robot joint velocities and accelerations are bounded as

$$|\dot{q}_i| \leq V_i > 0, \qquad |\ddot{q}_i| \leq A_i > 0, \qquad i = 1, 2,$$

while the velocity along the Cartesian path is bounded in norm as $\|\dot{\boldsymbol{p}}_d(t)\| \leq V_{c,max} > 0$. The robot is commanded by joint accelerations.

# Trajectory planning

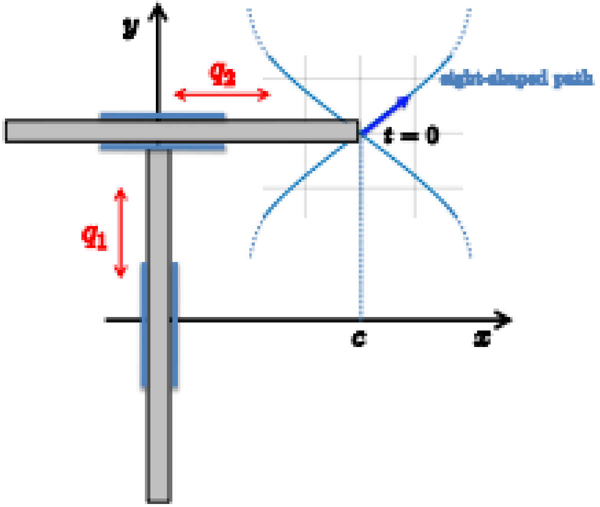


Figure 2: A 2P robot with the end-effector in the initial point of the desired trajectory at $t = 0$.

Give the symbolic expressions of the needed robot joint commands, and determine the maximum value $\omega_{max}$ of the angular frequency $\omega$ in (1) so that the robot motion satisfies all the constraints. Provide then the numerical value of $\omega_{max}$ using the following data: $a = 1$ [m], $b = 1.5$ [m], $c = 3$ [m], $V_1 = V_2 = 2$ [m/s], $V_{c,max} = 1.8$ [m/s], $A_1 = 2$ [m/s$^2$], $A_2 = 1.5$ [m/s$^2$].

# Trajectory planning

$$\dot{\boldsymbol{p}}_d(t) = \begin{pmatrix} 2a\omega\cos 2\omega t \\ b\omega\cos\omega t \end{pmatrix} = \begin{pmatrix} \dot{q}_2(t) \\ \dot{q}_1(t) \end{pmatrix},$$

and

$$\ddot{\boldsymbol{p}}_d(t) = \begin{pmatrix} -4a\omega^2\sin 2\omega t \\ -b\omega^2\sin\omega t \end{pmatrix} = \begin{pmatrix} \ddot{q}_2(t) \\ \ddot{q}_1(t) \end{pmatrix},$$

which are also the expressions of the robot joint commands. Moreover, the norm of (2) is

$$\|\dot{\boldsymbol{p}}_d(t)\| = \sqrt{4a^2\omega^2\cos^2 2\omega t + b^2\omega^2\cos^2\omega t}.$$

The bounds to be satisfied for all $t \in [0, 2\pi/\omega]$ are then

$$|\dot{q}_1| = |b\omega\cos\omega t| \le V_1 \quad\Rightarrow\quad \omega \le \frac{V_1}{b}, \qquad |\dot{q}_2| = |2a\omega\cos 2\omega t| \le V_2 \quad\Rightarrow\quad \omega \le \frac{V_2}{2a},$$

$$|\ddot{q}_1| = |-b\omega^2\sin\omega t| \le A_1 \quad\Rightarrow\quad \omega \le \sqrt{\frac{A_1}{b}}, \qquad |\ddot{q}_2| = |-4a\omega^2\sin 2\omega t| \le A_2 \quad\Rightarrow\quad \omega \le \sqrt{\frac{A_2}{4a}},$$

# Trajectory planning

$$\|\dot{\boldsymbol{p}}_d(t)\| = \omega\sqrt{4a^2\cos^2 2\omega t + b^2\cos^2\omega t} \le V_{c,max} \quad \Rightarrow \quad \omega \le \frac{V_{c,max}}{\sqrt{4a^2+b^2}}.$$

Therefore, the maximum feasible value of $\omega$ is

$$\omega_{max} = \min\left(\frac{V_1}{b}, \frac{V_2}{2a}, \sqrt{\frac{A_1}{b}}, \sqrt{\frac{A_2}{4a}}, \frac{V_{c,max}}{\sqrt{4a^2+b^2}}\right).$$

# Trajectory planning

Plan a cubic spline trajectory q(t) that interpolates the following data at given time instants:

$$t_1 = 1,\ q(t_1) = 45^\circ,\ t_2 = 2,\ q(t_2) = 90^\circ,\ t_3 = 2.5,\ q(t_3) = -45^\circ,\ t_4 = 4,\ q(t_4) = 45^\circ$$

starting with $\dot{q}(t_1) = 0$ and arriving with $\dot{q}(t_4) = 0$.

- Give an expression and the associated numerical values of the coefficients of each cubic polynomial.
- Find the maximum (absolute) values attained by the velocity $\dot{q}(t)$ and the acceleration $\ddot{q}(t)$ over the whole motion interval [t1, t4], as well as the time instants at which these occur.
- Check if the following bounds are satisfied throughout the motion, $|\ddot{q}(t)| \leq A_{max} = 1000^\circ/s^2$, and, if needed, determine the minimum uniform scaling factor for the trajectory so that feasibility is recovered.

# Trajectory planning

Using time normalization, the three cubic tracts of the interpolating spline are conveniently defined as

$$q_A(\tau_A) = q_1 + a_1\tau_A + a_2\tau_A^2 + a_3\tau_A^3, \qquad \tau_A = \frac{t - t_1}{t_2 - t_1} \in [0,1], \quad t \in [t_1, t_2]$$

$$q_B(\tau_B) = q_2 + b_1\tau_B + b_2\tau_B^2 + b_3\tau_B^3, \qquad \tau_B = \frac{t - t_2}{t_3 - t_2} \in [0,1], \quad t \in [t_2, t_3]$$

$$q_C(\tau_C) = q_3 + c_1\tau_C + c_2\tau_C^2 + c_3\tau_C^3, \qquad \tau_C = \frac{t - t_3}{t_4 - t_3} \in [0,1], \quad t \in [t_3, t_4],$$

with the nine coefficients $a_1, \ldots, c_3$ determined by satisfying the nine boundary conditions

$$\begin{aligned} q_A(1) &= q_2, \\ q_B(1) &= q_3, \\ q_C(1) &= q_4, \end{aligned} \qquad \begin{aligned} \dot{q}_A(0) &= 0, & \dot{q}_A(1) &= \dot{q}_B(0)\,[= v_2\,], & \ddot{q}_A(1) &= \ddot{q}_B(0), \\ \dot{q}_C(1) &= 0, & \dot{q}_B(1) &= \dot{q}_C(0)\,[= v_3\,], & \ddot{q}_B(1) &= \ddot{q}_C(0). \end{aligned}$$

# Trajectory planning

$$a_1 = 0, \qquad a_2 = 3(q_2 - q_1) - v_2(t_2 - t_1), \qquad a_3 = v_2(t_2 - t_1) - 2(q_2 - q_1),$$

and thus

$$\ddot{q}_A(1) = \frac{2a_2 + 6a_3}{(t_2 - t_1)^2} = \frac{4v_2}{t_2 - t_1} - \frac{6(q_2 - q_1)}{(t_2 - t_1)^2}.$$

Similarly, for the cubic B

$$b_1 = v_2(t_3 - t_2), \quad b_2 = 3(q_3 - q_2) - (2v_2 + v_3)(t_3 - t_2), \quad b_3 = -2(q_3 - q_2) + (v_2 + v_3)(t_3 - t_2),$$

and thus

$$\ddot{q}_B(0) = \frac{2b_2}{(t_3 - t_2)^2} = \frac{6(q_3 - q_2)}{(t_3 - t_2)^2} - \frac{4v_2 + 2v_3}{t_3 - t_2}$$

and

$$\ddot{q}_B(1) = \frac{2b_2 + 6b_3}{(t_3 - t_2)^2} = \frac{2v_2 + 4v_3}{t_3 - t_2} - \frac{6(q_3 - q_2)}{(t_3 - t_2)^2}.$$

Finally, for the cubic C

$$c_1 = v_3(t_4 - t_3), \qquad c_2 = 3(q_4 - q_3) - 2v_3(t_4 - t_3), \qquad c_3 = v_3(t_4 - t_3) - 2(q_4 - q_3),$$

and thus

$$\ddot{q}_C(0) = \frac{2c_2}{(t_4 - t_3)^2} = \frac{6(q_4 - q_3)}{(t_4 - t_3)^2} - \frac{4v_3}{t_4 - t_3}.$$

Imposing continuity of the acceleration at the internal knots

$$\ddot{q}_A(1) = \ddot{q}_B(0), \qquad \ddot{q}_B(1) = \ddot{q}_C(0),$$

# Trajectory planning

and using eqs. (28), (30–31) and (33), leads to the linear system of equations

$$\boldsymbol{A}\begin{pmatrix} v_2 \\ v_3 \end{pmatrix} = \boldsymbol{b},$$

with[1]

$$\boldsymbol{A} = \begin{pmatrix} 2(t_3 - t_1) & (t_2 - t_1) \\ (t_4 - t_3) & 2(t_4 - t_2) \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} 3(q_3 - q_2)\dfrac{t_2 - t_1}{t_3 - t_2} + 3(q_2 - q_1)\dfrac{t_3 - t_2}{t_2 - t_1} \\ 3(q_4 - q_3)\dfrac{t_3 - t_2}{t_4 - t_3} + 3(q_3 - q_2)\dfrac{t_4 - t_3}{t_3 - t_2}. \end{pmatrix}.$$

Replacing the numerical data (degrees are used everywhere here), the system is solved as

$$\begin{pmatrix} v_2 \\ v_3 \end{pmatrix} = \boldsymbol{A}^{-1}\boldsymbol{b} = \begin{pmatrix} -175.7143 \\ -215.3571 \end{pmatrix} \; [^\circ/\mathrm{s}],$$

and the coefficients (27), (29), and (32) of the three cubic polynomials take then the numerical values

$$a_0 = q_1 = 45, \quad a_1 = 0, \quad a_2 = 310.7143, \quad a_3 = -265.7143,$$

$$b_0 = q_2 = 90, \quad b_1 = -87.8571, \quad b_2 = -121.6071, \quad b_3 = 74.4643,$$

$$c_0 = q_3 = -45, \quad c_1 = -323.0357, \quad c_2 = 916.0714, \quad c_3 = -503.0357.$$

# Trajectory planning

$$A_1 = \ddot{q}(t_1) = \ddot{q}_A(0) = \frac{2a_2}{(t_2 - t_1)^2} = 621.4286,$$

$$A_2 = \ddot{q}(t_2) = \ddot{q}_B(0) = \frac{2b_2}{(t_3 - t_2)^2} = -972.8571,$$

$$A_3 = \ddot{q}(t_3) = \ddot{q}_C(0) = \frac{2c_2}{(t_4 - t_3)^2} = 814.2857,$$

$$A_4 = \ddot{q}(t_4) = \ddot{q}_C(1) = \frac{2c_2 + 6c_3}{(t_4 - t_3)^2} = -527.1429.$$

As a result, none of the (absolute) values exceeds the limit of $A_{\max} = 1000\,^\circ/\mathrm{s}^2$.

# Thank you for your Attention!!!