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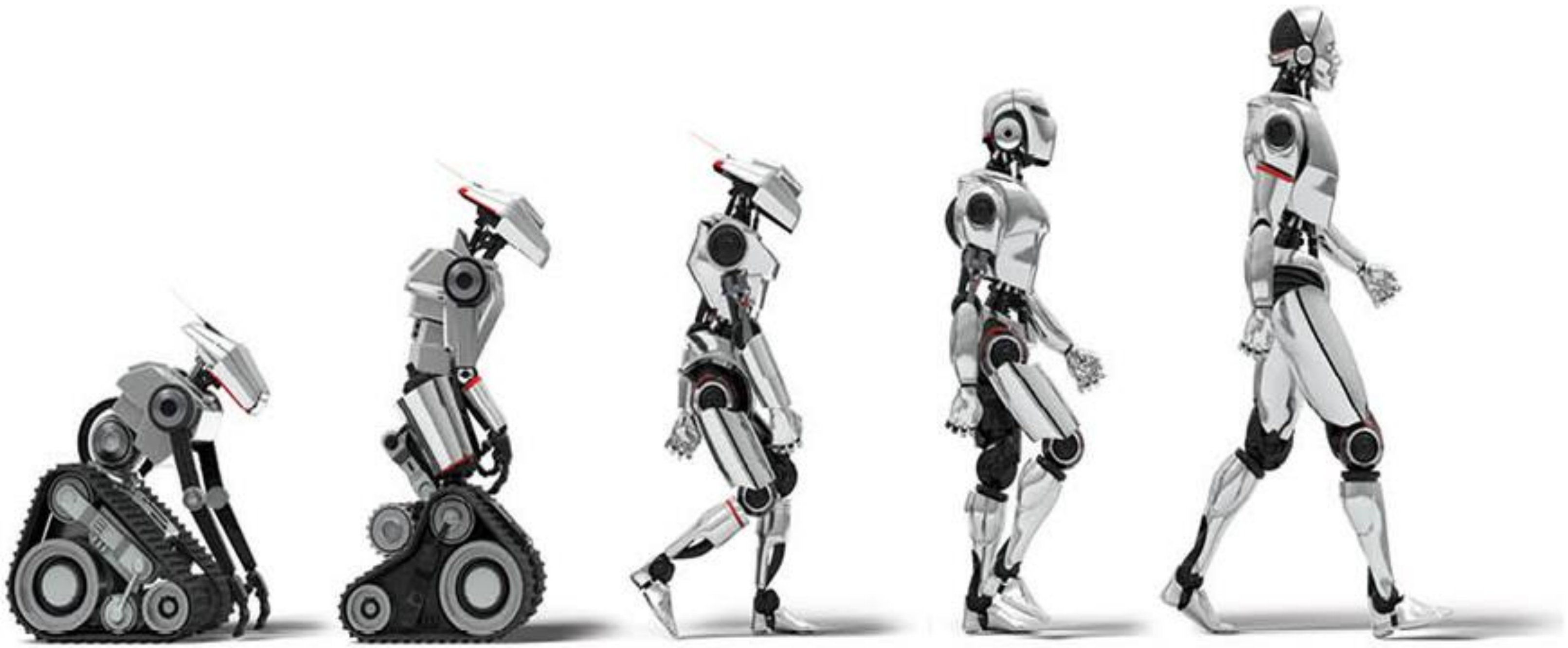
# Exercises Robotics 9



# Rotation Matrix

We are given an incomplete time-varying rotation matrix from frame 0 to frame 1:

$$ {}^{0}\boldsymbol{R}_1(t) = \begin{pmatrix} \cos t & a(t) & b(t) \\ \sin t & \dfrac{k(t)}{\sqrt{2}} \cos t & c(t) \\ 0 & -\dfrac{k(t)}{\sqrt{2}} \sin t & d(t) \end{pmatrix}. $$

Determine the expressions of $a(t)$, $b(t)$, $c(t)$, $d(t)$, and $k(t)$ in a consistent way.

# Rotation Matrix

We need to impose orthonormality conditions to the columns of ${}^0\boldsymbol{R}_1(t)$ and check finally that $\det {}^0\boldsymbol{R}_1(t) = +1$, for all times $t$. The first column $\boldsymbol{r}_1$ is already of unitary norm. For the second column $\boldsymbol{r}_2$, we need to impose the unit norm condition

$$\|\boldsymbol{r}_2\|^2 = a^2(t) + \frac{k^2(t)\cos^2 t}{2} + \frac{k^2(t)\sin^2 t}{2} = a^2(t) + \frac{k^2(t)}{2} = 1 \tag{1}$$

and the condition of orthogonality $\boldsymbol{r}_2 \perp \boldsymbol{r}_1$

$$a(t)\cos t + \frac{k(t)\cos t}{\sqrt{2}}\sin t = 0.$$

The latter provides $a(t) = -k(t)\sin t/\sqrt{2}$. Substituting in (1) yields

$$\frac{k^2(t)\sin^2 t}{2} + \frac{k^2(t)}{2} = 1 \qquad \Rightarrow \qquad k(t) = \pm\sqrt{\frac{2}{1+\sin^2 t}}. \tag{2}$$

Therefore, the second column of ${}^0\boldsymbol{R}_1(t)$ is

$$\boldsymbol{r}_2 = \begin{pmatrix} \dfrac{\mp\sin t}{\sqrt{1+\sin^2 t}} & \dfrac{\pm\cos t}{\sqrt{1+\sin^2 t}} & \dfrac{\mp\sin t}{\sqrt{1+\sin^2 t}} \end{pmatrix}^T. \tag{3}$$

Similarly, for the third column $\boldsymbol{r}_3$, we impose first the orthogonality $\boldsymbol{r}_3 \perp \boldsymbol{r}_1$

$$b(t)\cos t + c(t)\sin t = 0 \qquad \Rightarrow \qquad b(t) = \alpha(t)\sin t, \quad c(t) = -\alpha(t)\cos t. \tag{4}$$

# Rotation Matrix

Using (3) and (4), we impose next the orthogonality $\boldsymbol{r}_3 \perp \boldsymbol{r}_2$ as[1]

$$\alpha(t)\,\frac{\sin^2 t}{\sqrt{1+\sin^2 t}} + \alpha(t)\,\frac{\cos^2 t}{\sqrt{1+\sin^2 t}} + d(t)\,\frac{\sin t}{\sqrt{1+\sin^2 t}} = 0 \quad \Rightarrow \quad \alpha(t) = -d(t)\sin t.$$

Finally, the unit norm condition provides

$$\|\boldsymbol{r}_3\|^2 = 1 \quad \Rightarrow \quad d^2(t)\left(\sin^4 t + \sin^2 t\cos^2 t + 1\right) = 1 \quad \Rightarrow \quad d(t) = \frac{\pm 1}{\sqrt{1+\sin^2 t}}. \tag{5}$$

The uncertainty left in the signs of $k(t)$ and $d(t)$, respectively in eq. (2) and eq. (5), is eliminated by imposing the determinant of ${}^0\boldsymbol{R}_1(t)$ to be equal to $+1$. This holds true when choosing either both positive signs for $k(t)$ and $d(t)$, or both negative. The first solution is

$${}^0\boldsymbol{R}_1(t) = \begin{pmatrix} \cos t & -\dfrac{\sin t}{\sqrt{1+\sin^2 t}} & -\dfrac{\sin^2 t}{\sqrt{1+\sin^2 t}} \\ \sin t & \dfrac{\cos t}{\sqrt{1+\sin^2 t}} & \dfrac{\sin t\cos t}{\sqrt{1+\sin^2 t}} \\ 0 & -\dfrac{\sin t}{\sqrt{1+\sin^2 t}} & \dfrac{1}{\sqrt{1+\sin^2 t}} \end{pmatrix}, \tag{6}$$

and corresponds to the case when ${}^0\boldsymbol{R}_1(0) = \boldsymbol{I}$. The second solution is as in (6), but with each element of the second and third column having the opposite sign.

# Inverse Kinematics

The table of Denavit-Hartenberg parameters of a 2-dof robot is:

| $i$ | $\alpha_i$ | $a_i$ | $d_i$ | $\theta_i$ |
|---|---|---|---|---|
| 1 | $\pi/2$ | 0 | 0 | $q_1$ |
| 2 | 0 | 0 | $q_2$ | 0 |

The two joints have a range limitation: $|q_1| \leq 120°$ and $|q_2| \leq 2$ [m]. Determine all feasible inverse kinematics solutions, if any, when the origin of frame 2 needs to be placed at ${}^0\boldsymbol{p} = (-1, 1)$ [m].

# Inverse Kinematics

The given table of parameters refers to the planar RP robot in Fig. 3, where the associated Denavit-Hartenberg frames are also shown. Please note the definition of the first joint angle $q_1$, which differs from what one may expect (there is an additional $\pi/2$ with respect to the second link orientation).

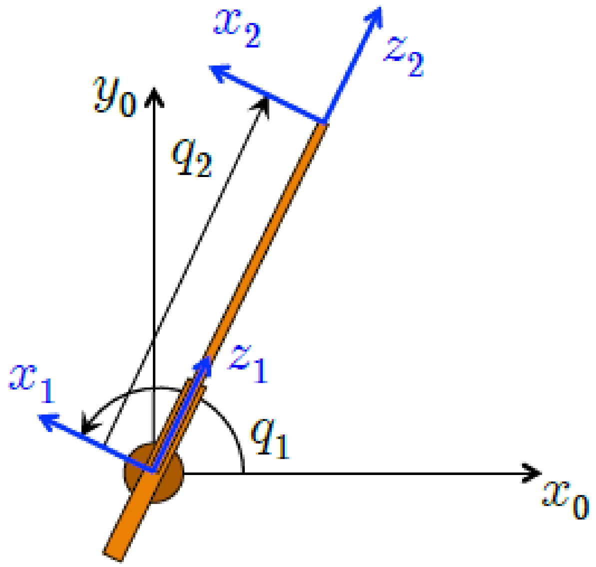



Figure 3: The RP robot, with its Denavit-Hartenberg frames and joint coordinates

The direct kinematics for the position $\boldsymbol{p}$ of the origin of frame 2 is then

$$\boldsymbol{p} = \begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} q_2 \sin q_1 \\ -q_2 \cos q_1 \end{pmatrix}.$$

# Inverse Kinematics

Out of the singularity ($q_2 \neq 0 \Leftrightarrow \boldsymbol{p} \neq 0$), the two solutions of the inverse kinematics are analytically found as

$$q_2 = \pm \|\boldsymbol{p}\| = \pm \sqrt{p_x^2 + p_y^2}, \qquad q_1 = \mathrm{ATAN2}\left\{\frac{p_x}{q_2}, -\frac{p_y}{q_2}\right\}. \tag{7}$$

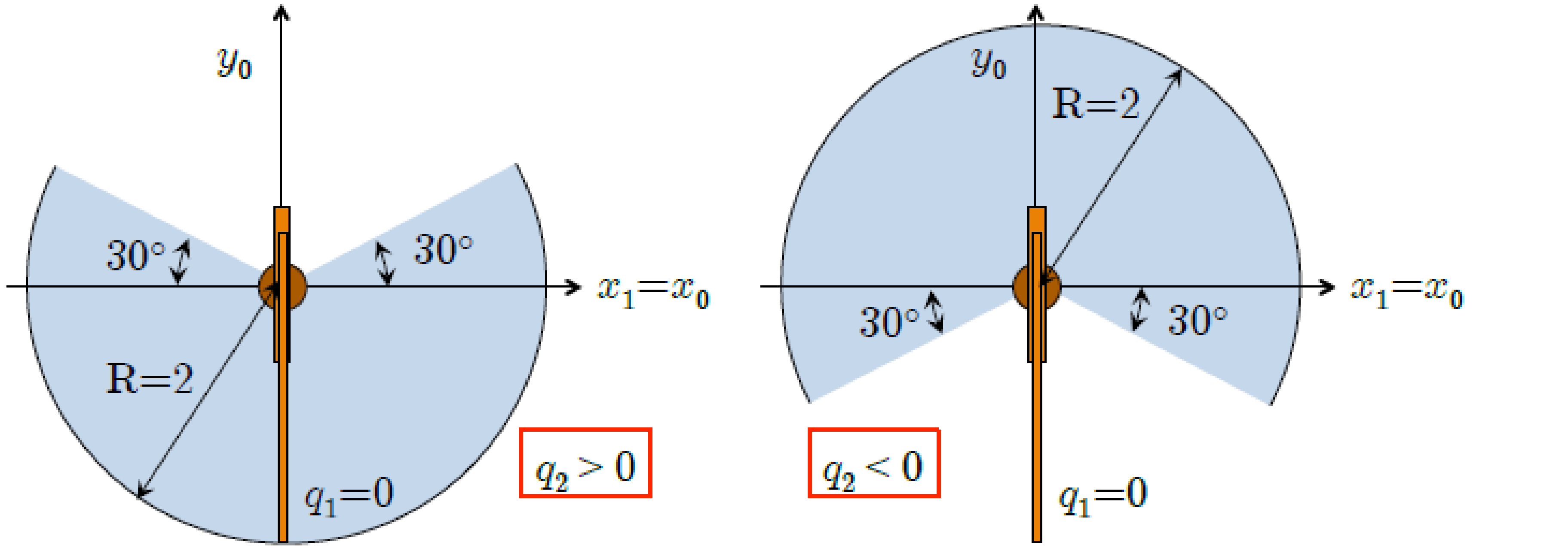



Figure 4: Robot workspace for $|q_1| \leq 120°$, $|q_2| \leq 2$, shown when $q_2 > 0$ (left) and $q_2 < 0$ (right)

# Inverse Kinematics

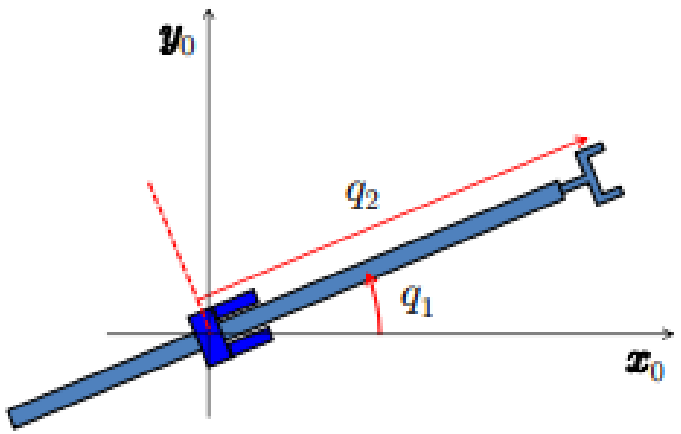



Figure 1: A RP planar robot, with the definition of the joint variables.

The RP robot shown in Fig. 1 starts from rest at time $t = 0$ in the configuration $\boldsymbol{q}(0) = (0, 1)$ [rad; m] and moves under the action of the following discontinuous joint acceleration commands for a time $T = 2$ [s]:

$$\ddot{q}_1(t) = \begin{cases} A_1 = 2 \text{ [rad/s}^2\text{]}, & t \in [0, T/4], \\ 0, & t \in [T/4, 3T/4], \\ -A_1 = -2 \text{ [rad/s}^2\text{]}, & t \in [3T/4, T]; \end{cases} \qquad \ddot{q}_2(t) = \begin{cases} -A_2 = -0.5 \text{ [m/s}^2\text{]}, & t \in [0, T/2], \\ A_2 = 0.5 \text{ [m/s}^2\text{]}, & t \in [T/2, T]. \end{cases}$$

a. Plot the time profiles of $q_i(t)$, $\dot{q}_i(t)$ and $\ddot{q}_i(t)$, for $i = 1, 2$.

b. Does the robot cross a singularity during this motion?

c. Compute the mid time configuration $\boldsymbol{q}(T/2)$ and the final configuration $\boldsymbol{q}(T)$ reached in this motion. Sketch the robot in these two configurations, as well as in the initial one.

# Thank you for your Attention!!!