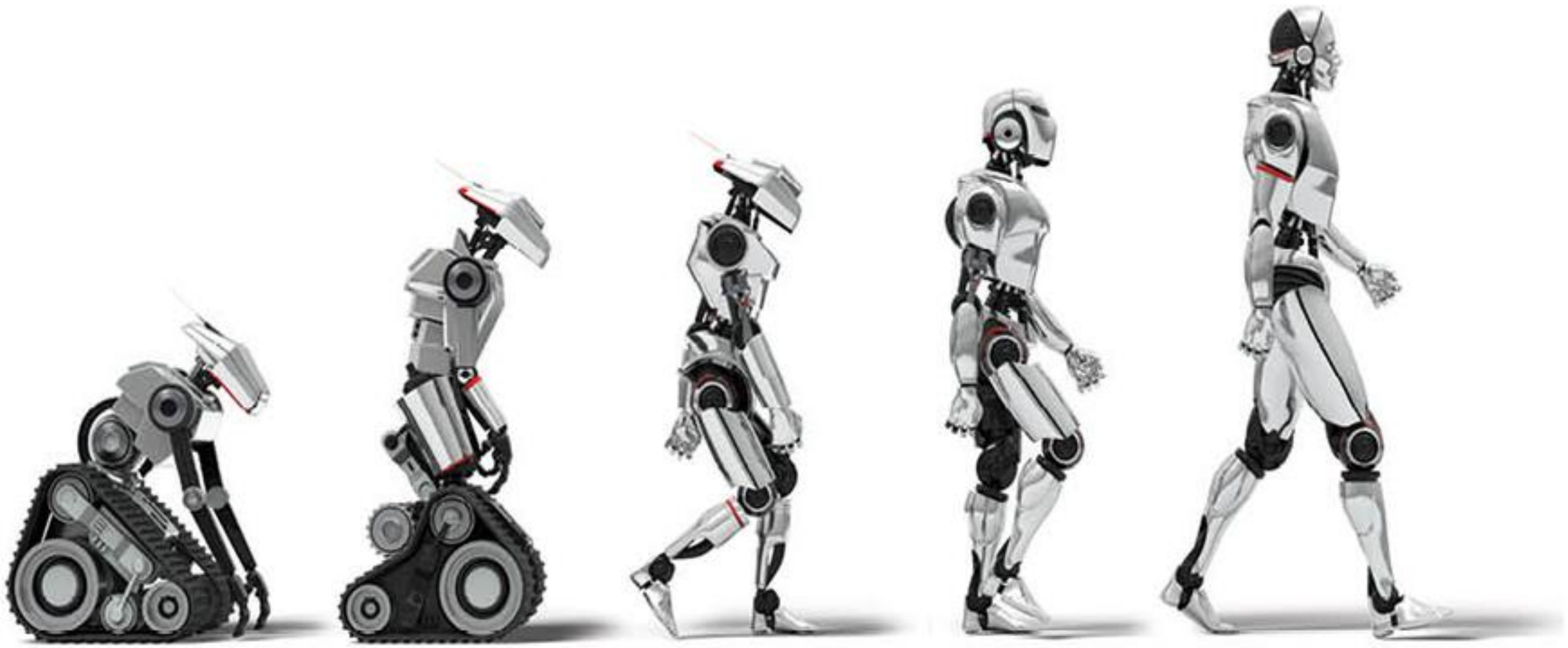
UNIVERSITÄT
HEIDELBERG
ZUKUNFT
SEIT 1386

**Universität Heidelberg**

*Fakultät für Physik und Astronomie*

# Exercises Robotics 11



# Jacobian

The kinematics of a 3R robot is defined by the following Denavit-Hartenberg table (units in [m] or [rad]):

| $i$ | $\alpha_i$ | $a_i$ | $d_i$ | $\theta_i$ |
|---|---|---|---|---|
| 1 | $\pi/2$ | 0 | $d_1 = 5$ | $q_1$ |
| 2 | 0 | $a_2 = 4$ | 0 | $q_2$ |
| 3 | 0 | $a_3 = 3$ | 0 | $q_3$ |

Determine the $3 \times 3$ linear part of the geometric Jacobian $\boldsymbol{J}(\boldsymbol{q})$ of this robot. When the robot is in the configuration $\boldsymbol{q}_0 = (\pi/2, \pi/4, \pi/2)$ [rad] and has a joint velocity $\dot{\boldsymbol{q}}_0 = (1, 2, -2)$ [rad/s], determine, if possible, a joint acceleration $\ddot{\boldsymbol{q}}$ that realizes a zero end-effector acceleration, i.e., $\ddot{\boldsymbol{p}} = \mathbf{0}$. *[Bonus: What if the second link parameter is changed to $a_2 = 3$?]*

# Jacobian

The end-effector position is obtained from the homogenous transformation matrices:

$$\boldsymbol{p}_H = \begin{pmatrix} \boldsymbol{p} \\ 1 \end{pmatrix} = {}^0\boldsymbol{A}_1(q_1)\left({}^1\boldsymbol{A}_2(q_2)\left({}^2\boldsymbol{A}_3(q_3)\begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix}\right)\right)$$

$$\Longrightarrow \qquad \boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} \cos q_1 \left(a_2 \cos q_2 + a_3 \cos(q_2 + q_3)\right) \\ \sin q_1 \left(a_2 \cos q_2 + a_3 \cos(q_2 + q_3)\right) \\ d_1 + a_2 \sin q_2 + a_3 \sin(q_2 + q_3) \end{pmatrix}. \tag{1}$$

Therefore, using the usual compact notation for trigonometric functions, we have

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -s_1 \left(a_2 c_2 + a_3 c_{23}\right) & -c_1 \left(a_2 s_2 + a_3 s_{23}\right) & -a_3 c_1 s_{23} \\ c_1 \left(a_2 c_2 + a_3 c_{23}\right) & -s_1 \left(a_2 s_2 + a_3 s_{23}\right) & -a_3 s_1 s_{23} \\ 0 & a_2 c_2 + a_3 c_{23} & a_3 c_{23} \end{pmatrix}. \tag{2}$$

The Jacobian is singular when[1]

$$\det \boldsymbol{J}(\boldsymbol{q}) = -a_2 a_3\, s_3 \left(a_2 c_2 + a_3 c_{23}\right) = 0. \tag{3}$$

# Jacobian

The end-effector acceleration $\ddot{\boldsymbol{p}}$ is computed as

$$\ddot{\boldsymbol{p}} = \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}. \tag{4}$$

Thus, in order to realize a zero end-effector acceleration, we need to set $\ddot{\boldsymbol{p}} = \boldsymbol{0}$ in (4) and solve for $\ddot{\boldsymbol{q}}$, or

$$\ddot{\boldsymbol{q}} = -\boldsymbol{J}^{-1}(\boldsymbol{q})\dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}. \tag{5}$$

Indeed, this solution is valid as long as the robot is out of singularities. To evaluate (5), we need first to derive the time derivative of the robot Jacobian. Let $\boldsymbol{J}_i(\boldsymbol{q})$ be the $i$th column of the Jacobian $\boldsymbol{J}(\boldsymbol{q})$, for $i = 1, 2, 3$. We compute

$$\begin{aligned}
\dot{\boldsymbol{J}}(\boldsymbol{q}) &= \frac{d\boldsymbol{J}(\boldsymbol{q})}{dt} \left( = \sum_{i=1}^{3} \frac{\partial \boldsymbol{J}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} \dot{\boldsymbol{q}} = \sum_{j=1}^{3} \frac{\partial \boldsymbol{J}(\boldsymbol{q})}{\partial q_j} \dot{q}_j \right) \\
&= \begin{pmatrix}
-c_1\dot{q}_1\left(a_2c_2 + a_3c_{23}\right) + s_1\left(a_2s_2\dot{q}_2 + a_3s_{23}(\dot{q}_2+\dot{q}_3)\right) & s_1\dot{q}_1\left(a_2s_2 + a_3s_{23}\right) - c_1\left(a_2c_2\dot{q}_2 + a_3c_{23}(\dot{q}_2+\dot{q}_3)\right) & a_3s_1\dot{q}_1s_{23} - a_3c_1c_{23}(\dot{q}_2+\dot{q}_3) \\
-s_1\dot{q}_1\left(a_2c_2 + a_3c_{23}\right) - c_1\left(a_2s_2\dot{q}_2 + a_3s_{23}(\dot{q}_2+\dot{q}_3)\right) & c_1\dot{q}_1\left(a_2s_2 + a_3s_{23}\right) - s_1\left(a_2c_2\dot{q}_2 + a_3c_{23}(\dot{q}_2+\dot{q}_3)\right) & -a_3c_1\dot{q}_1s_{23} - a_3s_1c_{23}(\dot{q}_2+\dot{q}_3) \\
0 & -\left(a_2s_2\dot{q}_2 - a_3s_{23}(\dot{q}_2+\dot{q}_3)\right) & -a_3s_{23}(\dot{q}_2+\dot{q}_3)
\end{pmatrix}.
\end{aligned} \tag{6}$$

# Jacobian

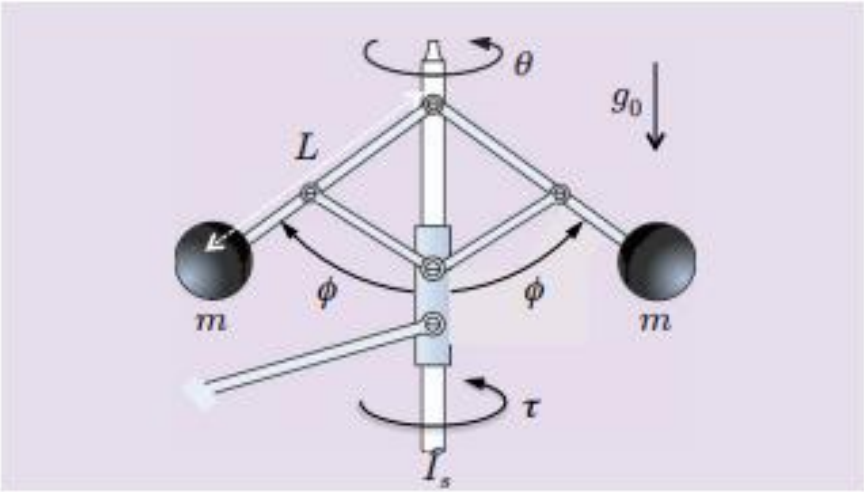
When the robot is in the configuration $\boldsymbol{q}_0 = (\pi/2, \pi/4, \pi/2)$ [rad] and has a joint velocity $\dot{\boldsymbol{q}}_0 = (1, 2, -2)$ [rad/s], evaluation of (2) and (6) gives

$$\boldsymbol{J}_0 = \boldsymbol{J}(\boldsymbol{q}_0) = \begin{pmatrix} -0.7071 & 0 & 0 \\ 0 & -4.9497 & -2.1213 \\ 0 & 0.7071 & -2.1213 \end{pmatrix}, \quad \dot{\boldsymbol{J}}_0 = \dot{\boldsymbol{J}}(\boldsymbol{q}_0)|_{\dot{\boldsymbol{q}}=\dot{\boldsymbol{q}}_0} = \begin{pmatrix} 5.6569 & 4.9497 & 2.1213 \\ -0.7071 & -5.6569 & 0 \\ 0 & -5.6569 & 0 \end{pmatrix}.$$

Since $\det \boldsymbol{J}_0 = -8.4853 \neq 0$,we can use eq. (5) for computing the joint acceleration $\ddot{\boldsymbol{q}}$ that realizes a zero end-effector acceleration:

$$\ddot{\boldsymbol{q}} = -\boldsymbol{J}_0^{-1}\left(\dot{\boldsymbol{J}}_0\dot{\boldsymbol{q}}_0\right) = -\begin{pmatrix} -1.4142 & 0 & 0 \\ 0 & -0.1768 & 0.1768 \\ 0 & -0.0589 & -0.4125 \end{pmatrix}\begin{pmatrix} 11.3137 \\ -12.0208 \\ -11.3137 \end{pmatrix} = \begin{pmatrix} 16 \\ -0.1250 \\ -5.3750 \end{pmatrix} [\text{rad/s}^2]. \quad (7)$$

# Lagrangian




Assume that:

- the main shaft has an inertia $I_s$ around its rotation axis
- the two balls have identical mass $m$ that is concentrated at the end of a link of length $L$
- the links and all other linkages have negligible masses
- a viscous friction torque with coefficient $f_v > 0$ is acting on the main shaft
- all other frictional effects are negligible.

Derive the complete dynamic model of this system using a Lagrangian formalism. Assuming knowledge of the geometric parameter $L$, provide a linear parametrization of the dynamics in terms of its dynamic coefficients.

Thank you for your Attention!!!