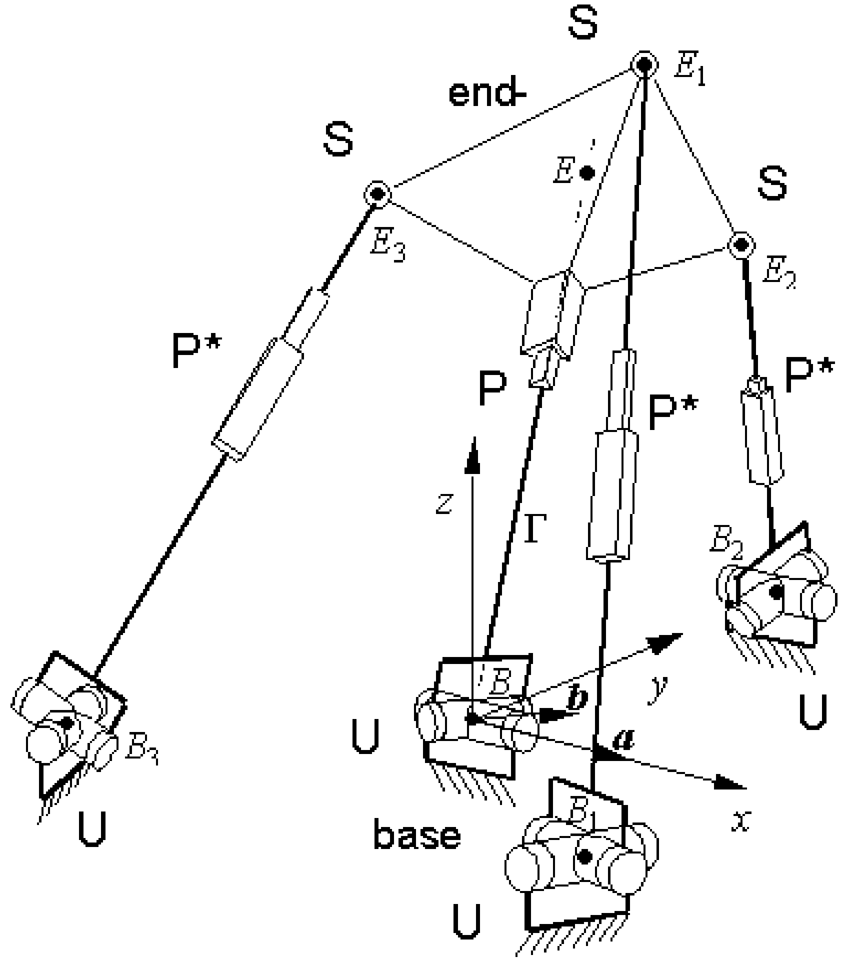
Robotics 1
Robot kinematics



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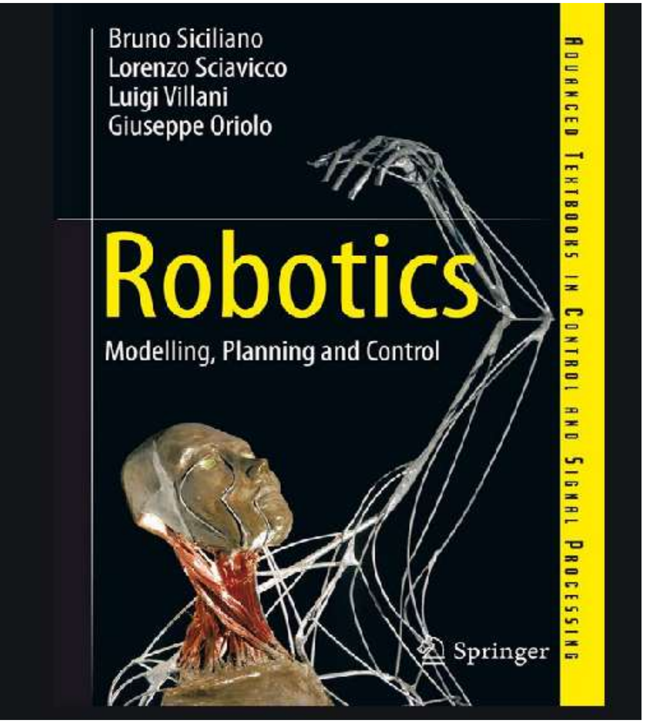
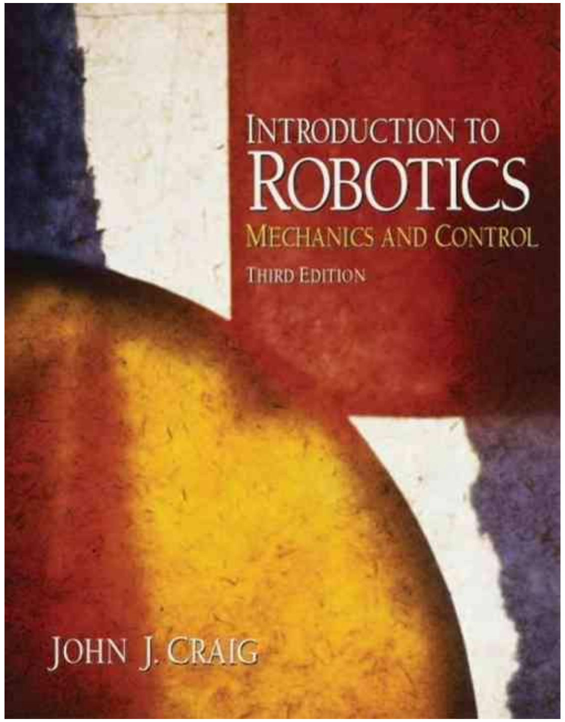
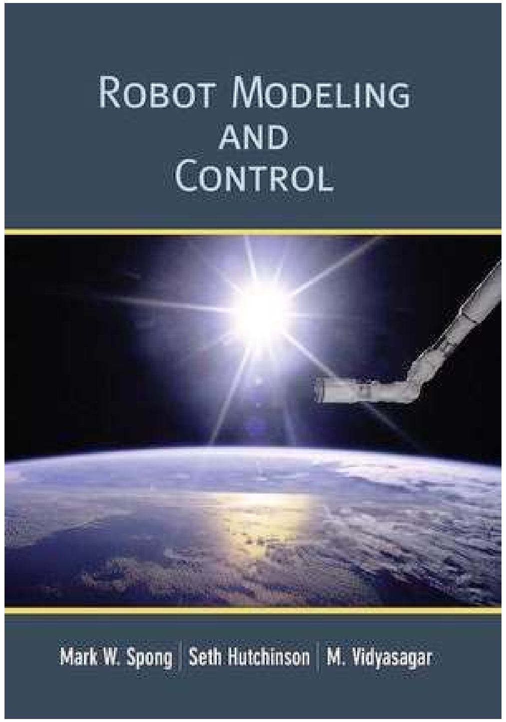
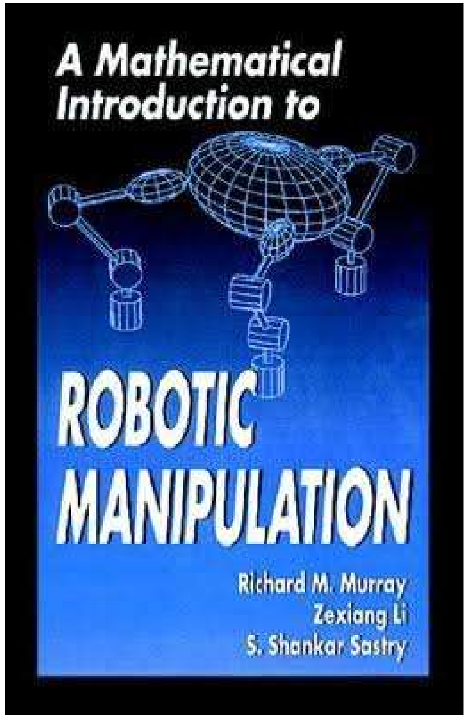
Advised books









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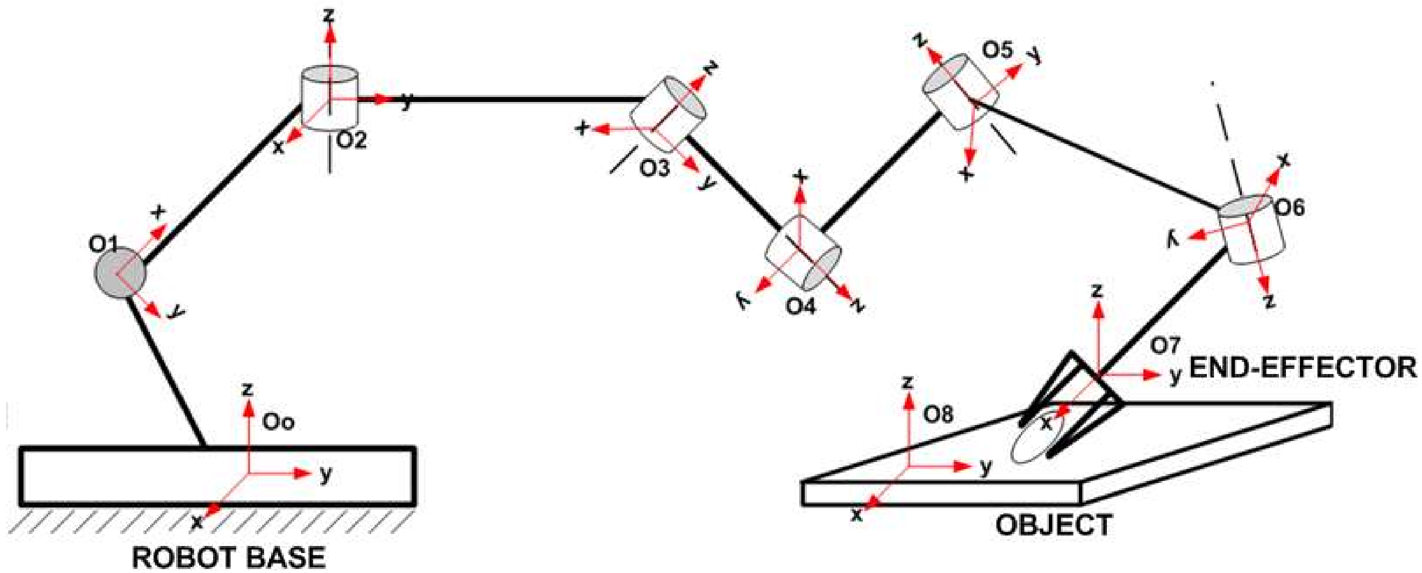
Robot Kinematics

- A large part of robot kinematics is concerned with the establishment of various coordinate systems to represent the positions and orientations of rigid objects and with transformations among these coordinate systems.
- Indeed, the geometry of three-dimensional space and of rigid motions plays a central role in all aspects of robotic manipulation

it is instructive to distinguish between the two fundamental approaches to geometric reasoning:
the synthetic approach reasons directly about geometric entities (e.g., points or lines),
the analytic approach represents these entities using coordinates or equations, and reasoning is performed via algebraic manipulations.

Robot Kinematics

First problem in programming robots is to describe the position of the «**end-effector**» in relation to a fixed frame usually called «**base**»



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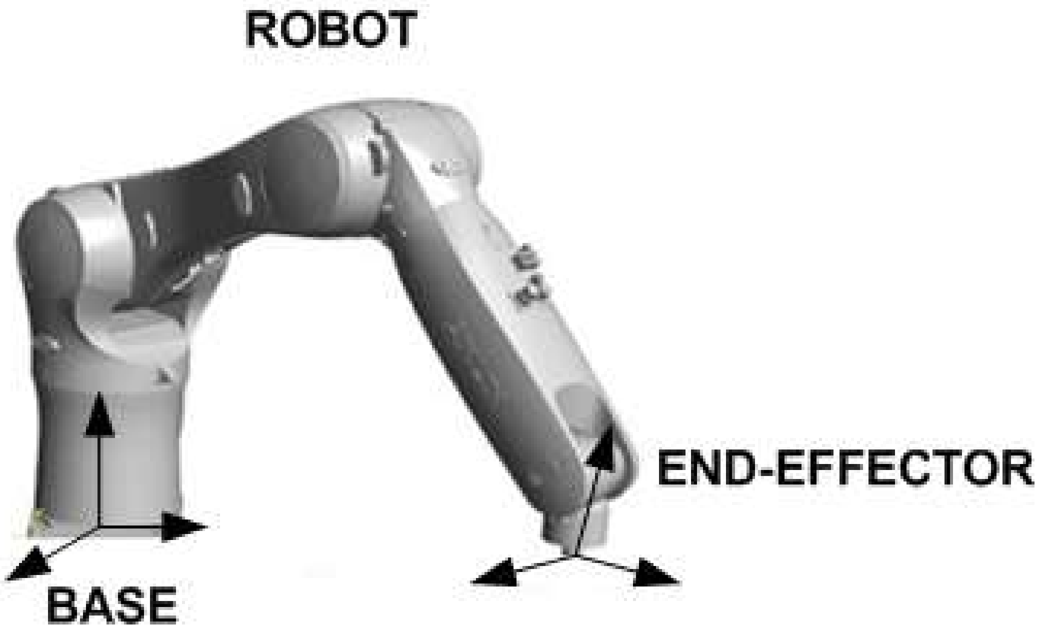
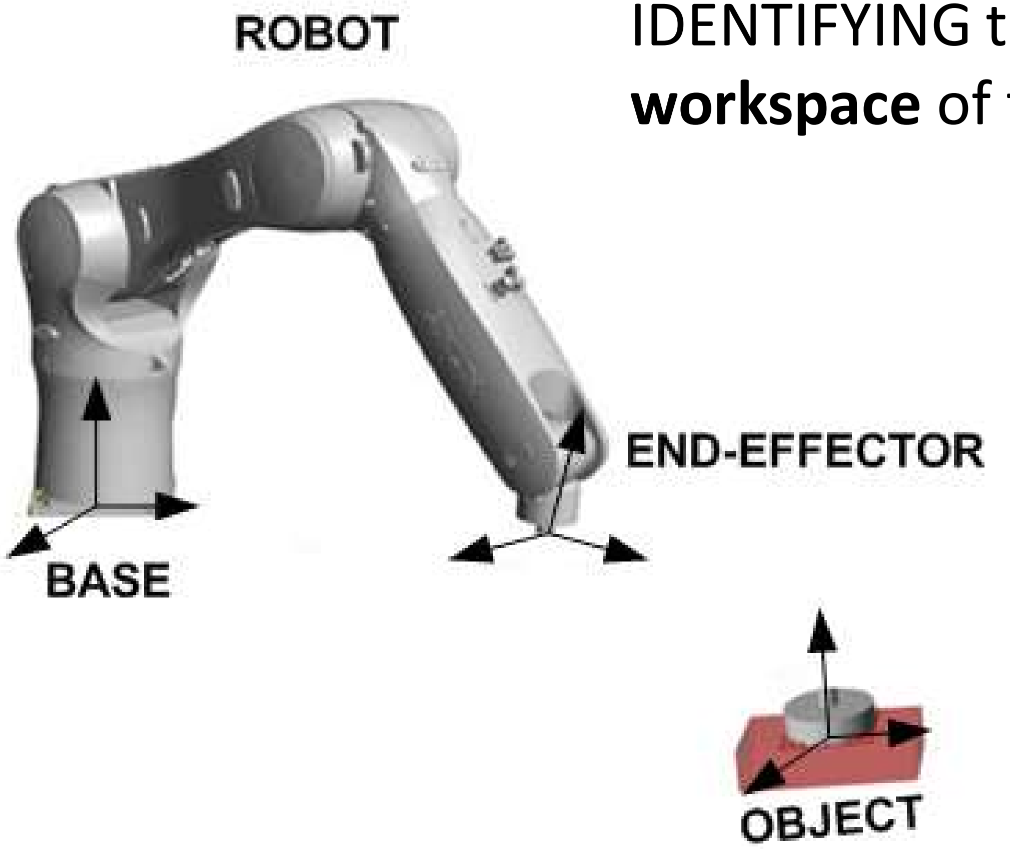
ROBOT KINEMATICS: STEP 1



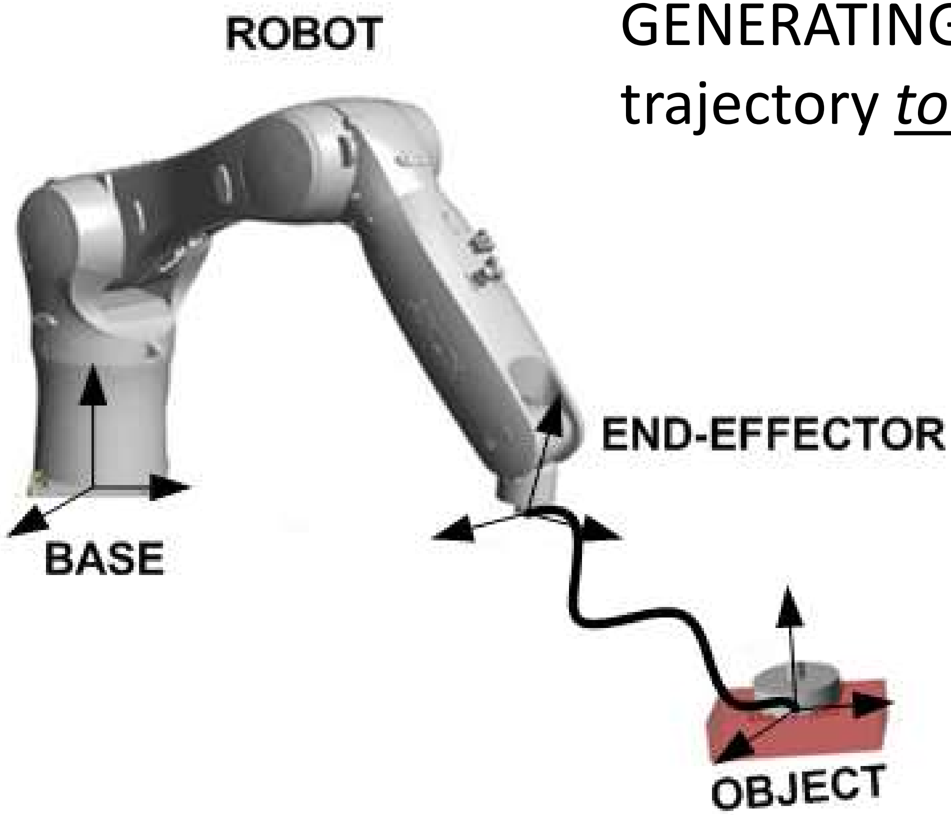
FORMULATING the position of the «**end-effector**» in relation to the «**base**»

Robot Kinematics: step 2



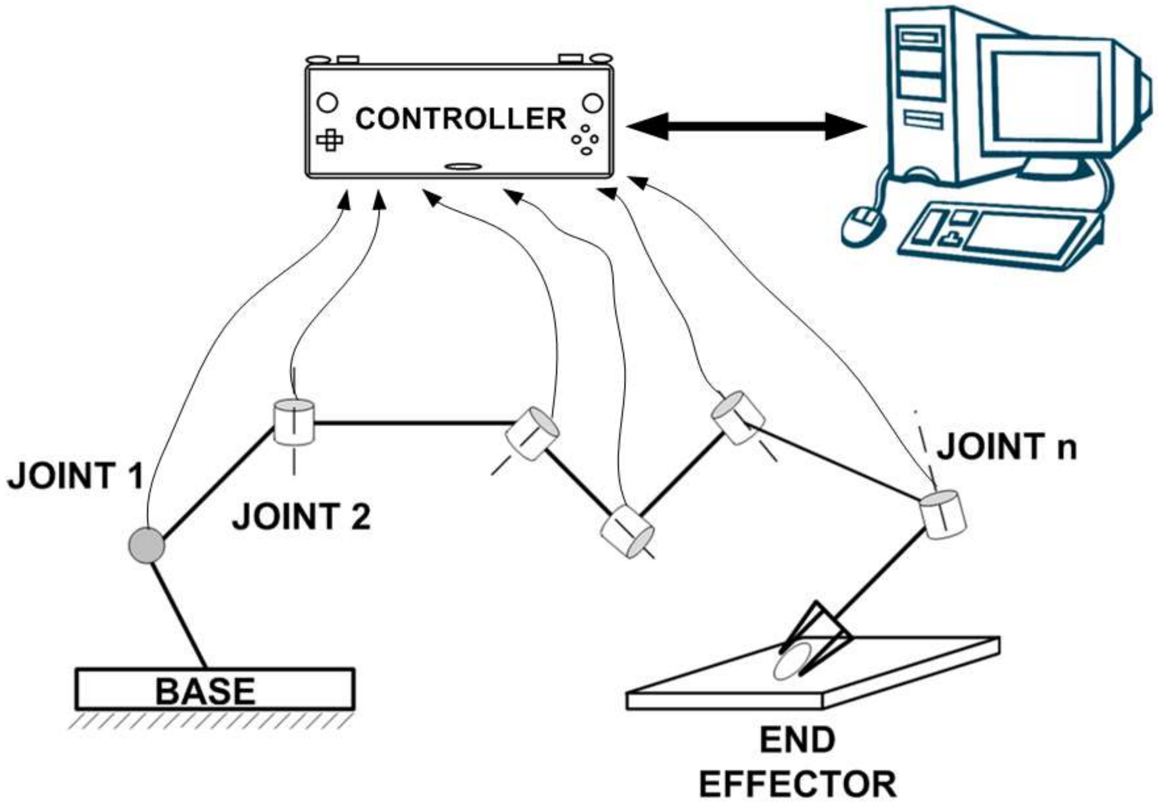
IDENTIFYING the task in the **workspace** of the robot

Robot Kinematics: step 3



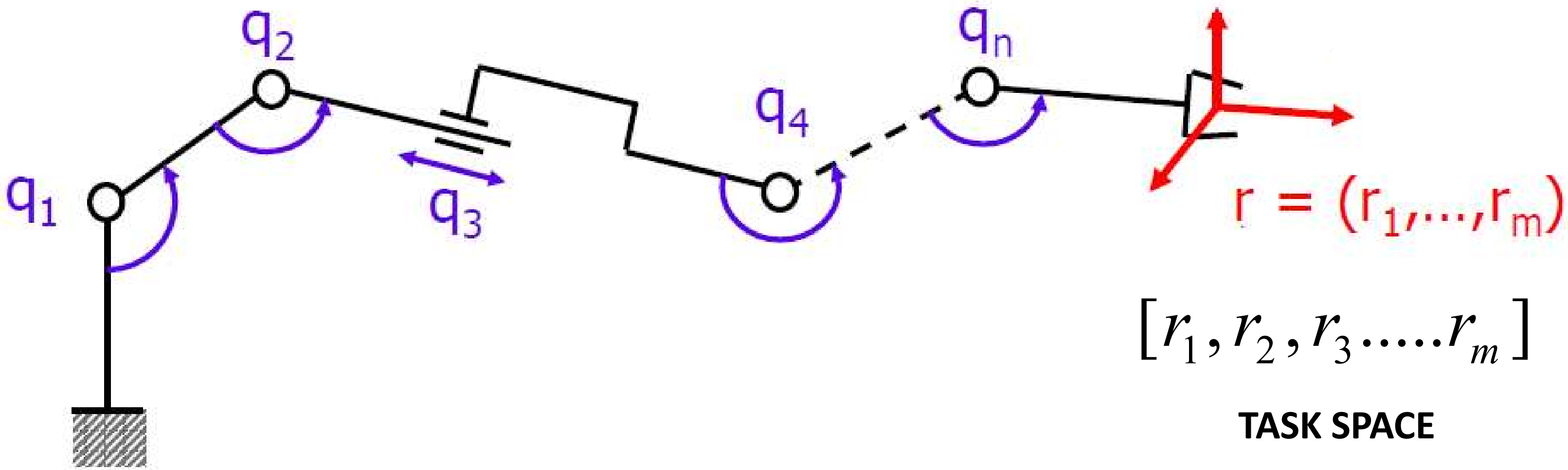
GENERATING the opportune trajectory *to accomplish* the task

kinematics



Robot joints are equipped with sensors (encoders or resolvers) feeding back their ***rotation*** to the central CPU

kinematics



$$[r_1, r_2, r_3 \ldots\ldots r_m]$$

TASK SPACE

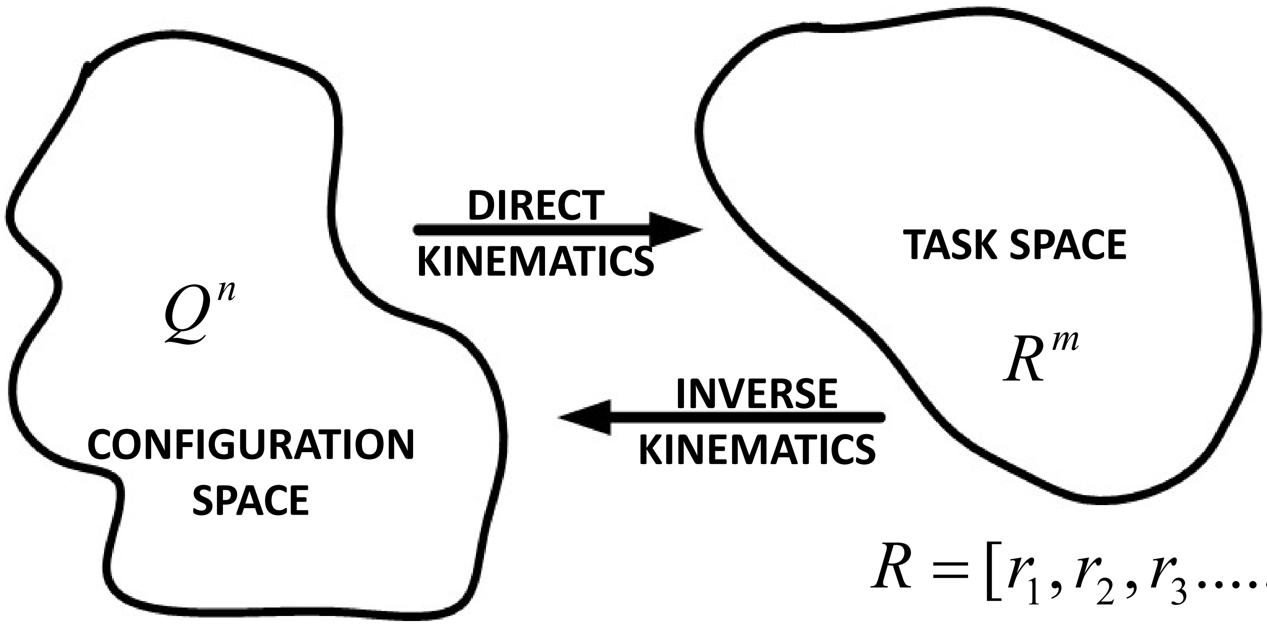
$$[q_1, q_2, q_3 \ldots\ldots q_n]$$

CONFIGURATION SPACE

How to relate the two SPACES?

$$[q_1, q_2, q_3 \ldots\ldots q_n] \Longleftrightarrow [r_1, r_2, r_3 \ldots\ldots r_m]$$

Kinematics



$$R = [r_1, r_2, r_3 .....r_m]$$

$$Q = [q_1, q_2, q_3 .....q_n]$$

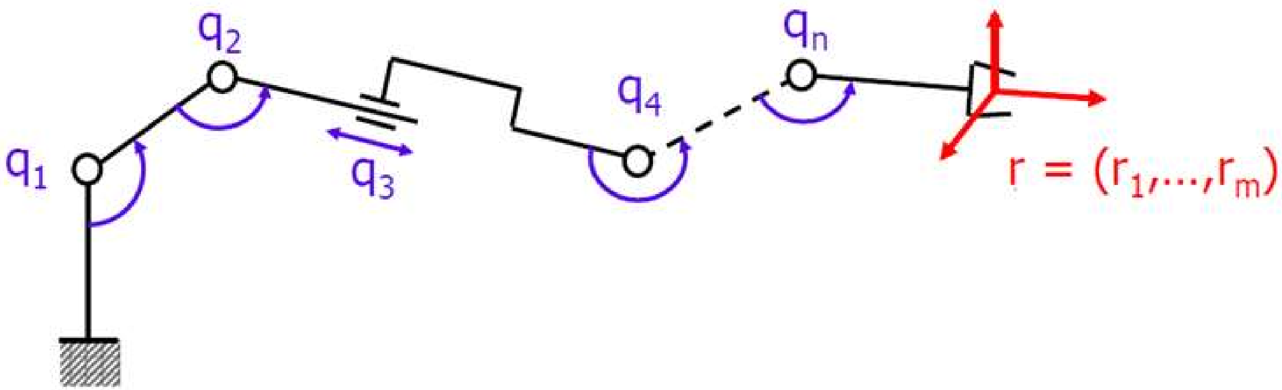
•The dimension of the *configuration space* must be **larger or equal** to the dimension of the *task space* $(n \geq m)$

•To ensure the existence of Kinematics solutions.

Forward Kinematics

- The process of finding the position/orientation of the end-effector **(r_1,…..r_m)** given a set of joint angles **(q_1,…..q_n)** is known as *forward kinematics*;

$$(r_1, r_2, r_3 .... r_m) = F(q_1, q_2, q_3 ..... q_n)$$



Forward Kinematics

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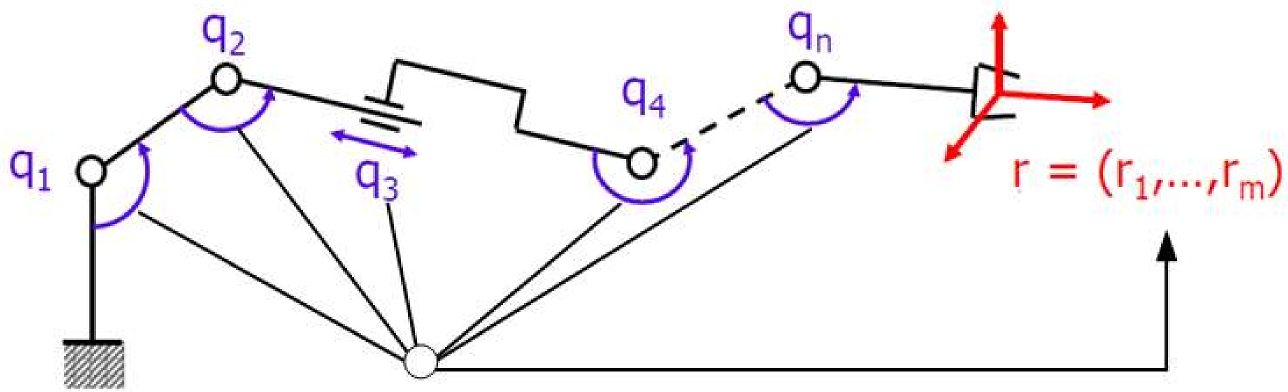
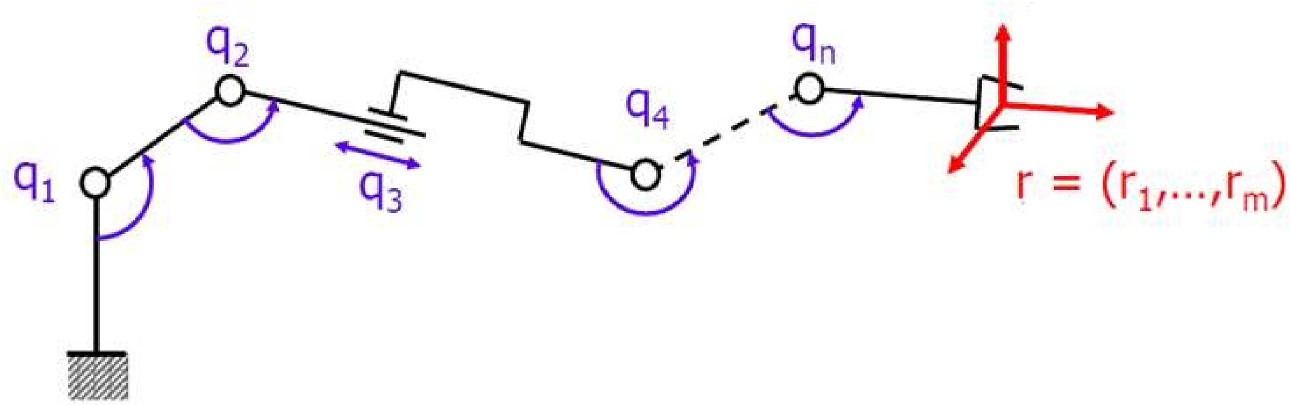
$$(r_1, r_2, r_3 \ldots . r_m) = F(q_1, q_2, q_3 \ldots .. q_n)$$



inverse Kinematics

The process of finding the **joint angles (q_1,.....q_n)** that realizes a given (desired) position/orientation of the end-effector **(r_1,.....r_m)** is known as <u>*inverse kinematics*</u>.

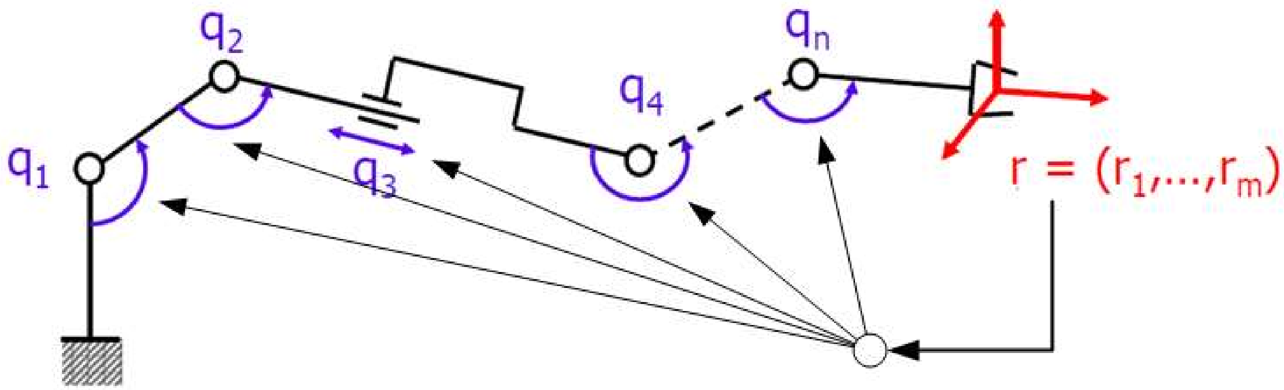
$$(q_1, q_2, q_3 ....., q_n) = G(r_1, r_2, r_3 ..... r_m)$$



inverse Kinematics

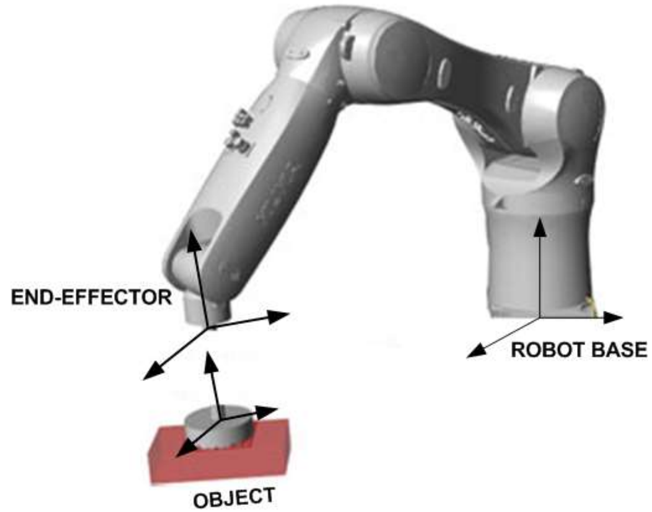
The process of finding the **joint angles (q_1,.....q_n)** that realizes a given (desired) position/orientation of the end-effector **(r_1,.....r_m)** is known as *inverse kinematics*.

$$(q_1, q_2, q_3 \ldots.., q_n) = G(r_1, r_2, r_3 \ldots.. r_m)$$



Kinematics: spatial description and transformation

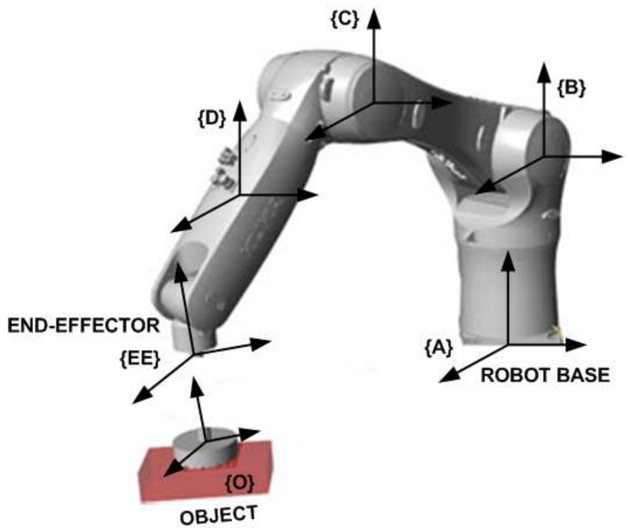
Robotic manipulation implies multiple actions:
•Moving tools
•Picking objects
•Assembling parts



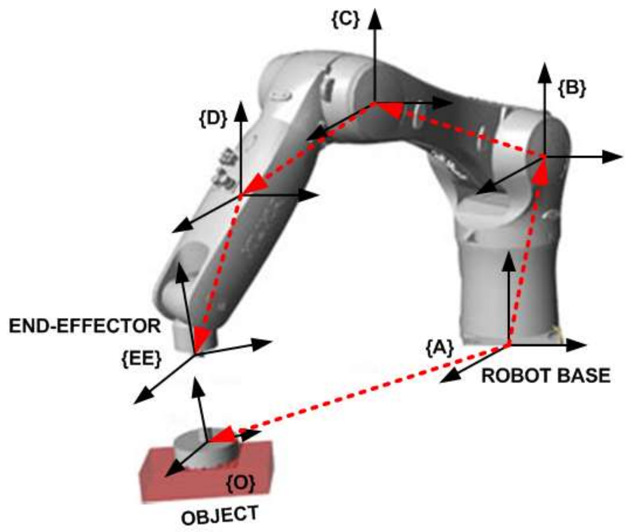
We must relate the kinematics of the **object** to be manipulated with the one of the **robotic manipulator**.

<u>Both the robot and the object must have Reference Frames</u>

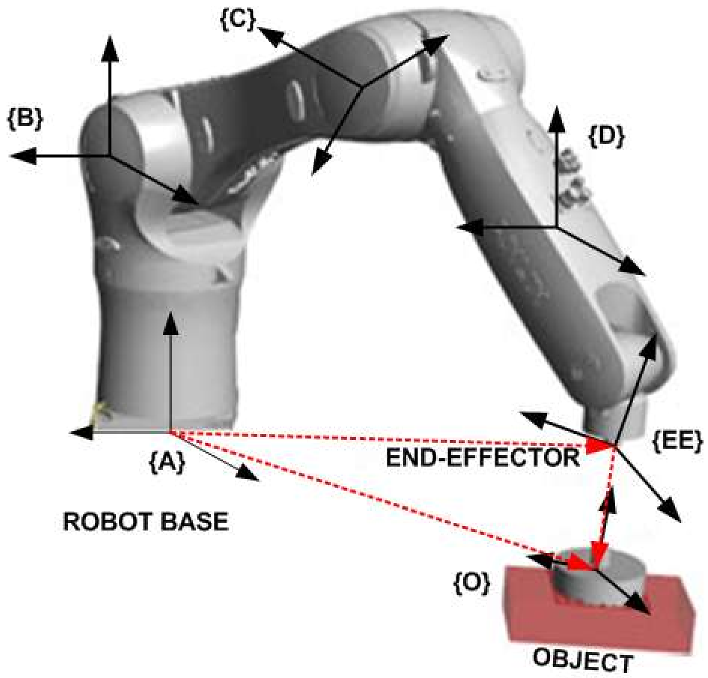
KINEMATICS: SPATIAL DESCRIPTION AND TRANSFORMATION

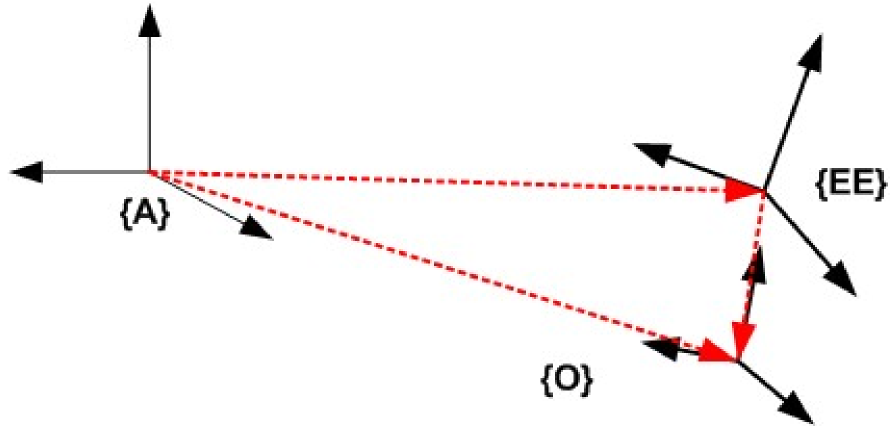


Coordinated Reference frames
- the robot
- the object



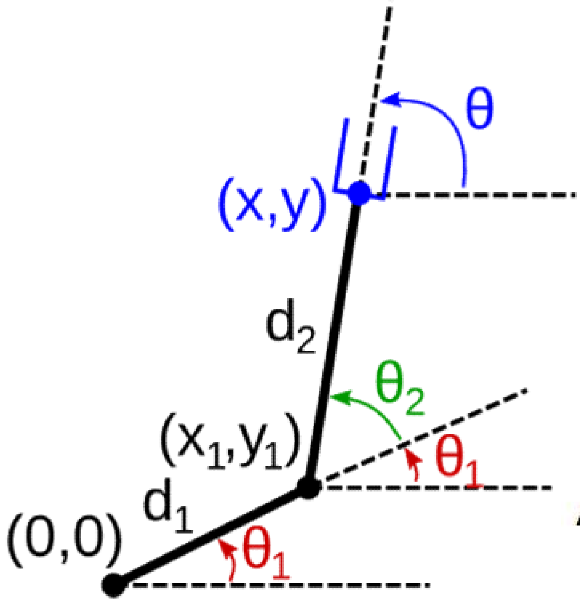
Finding the Transformations among the Reference frames


{C}
{B}
{D}
{EE}
END-EFFECTOR
{A}
ROBOT BASE
{O}
OBJECT


{A}
{EE}
{O}

Example: Forward Kinematics

Consider a planar manipulator with two revolute joints:



Given The *joint angles* : $\boldsymbol{q}:=(\theta_1,\theta_2)$

Unknown: The *position and the orientation* of the end-effector : $\boldsymbol{p}:=(x,y,\theta)$

Solution is The *forward kinematics* problem : $\boldsymbol{q}\mapsto\boldsymbol{p}$.

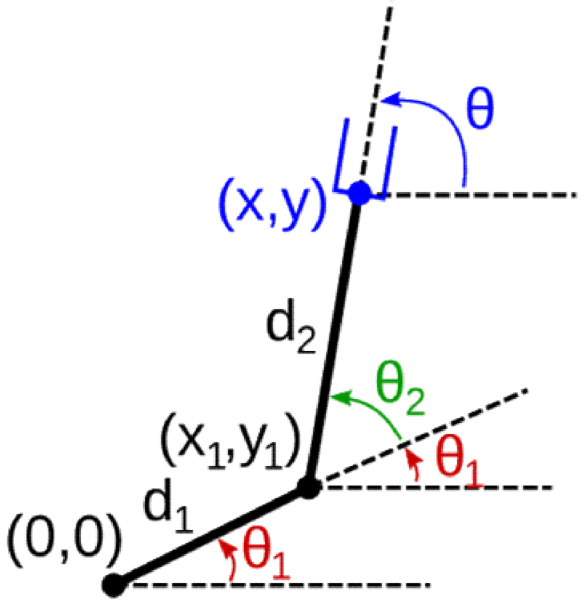
Applying simple trigonometry on the first link, one has

$$\begin{cases} x_1 &= d_1\cos(\theta_1) \\ y_1 &= d_1\sin(\theta_1) \end{cases}$$

By similar calculations on the second link, one obtains

$$\begin{cases} x &= x_1 + d_2\cos(\theta_1+\theta_2) &= d_1\cos(\theta_1) + d_2\cos(\theta_1+\theta_2) \\ y &= y_1 + d_2\sin(\theta_1+\theta_2) &= d_1\sin(\theta_1) + d_2\sin(\theta_1+\theta_2) \end{cases}$$

Example: Forward Kinematics



The *forward kinematics* problem : $\boldsymbol{q} \mapsto \boldsymbol{p}$.

$$\begin{cases} x &= d_1 \cos(\theta_1) + d_2 \cos(\theta_1 + \theta_2) \\ y &= d_1 \sin(\theta_1) + d_2 \sin(\theta_1 + \theta_2) \end{cases} .$$

Finally, the orientation of the manipulator is given by $\theta = \theta_1 + \theta_2$.

One has thus obtained the explicit formulae for the forward kinematics function FK.

Jacobian matrix

Definition

[Jacobian matrix]

The Jacobian matrix of the forward kinematics mapping at a given configuration $\boldsymbol{q}_0$ is defined by:

$$\boldsymbol{J}(\boldsymbol{q}_0) := \left.\frac{\partial \mathrm{FK}(\boldsymbol{q})}{\partial \boldsymbol{q}}\right|_{\boldsymbol{q}=\boldsymbol{q}_0}$$

In the case of the planar 2-DOF manipulator, one has

$$\boldsymbol{J}(\theta_1, \theta_2) = \begin{pmatrix} \frac{\partial x}{\partial \theta_1} & \frac{\partial x}{\partial \theta_2} \\ \frac{\partial y}{\partial \theta_1} & \frac{\partial y}{\partial \theta_2} \\ \frac{\partial \theta}{\partial \theta_1} & \frac{\partial \theta}{\partial \theta_2} \end{pmatrix} = \begin{pmatrix} -d_1 \sin(\theta_1) - d_2 \sin(\theta_1 + \theta_2) & -d_2 \sin(\theta_1 + \theta_2) \\ d_1 \cos(\theta_1) + d_2 \cos(\theta_1 + \theta_2) & d_2 \cos(\theta_1 + \theta_2) \\ 1 & 1 \end{pmatrix}$$

Jacobian matrix

Configuration space (n=2)

$$\boldsymbol{J}(\theta_1, \theta_2) = \begin{pmatrix} -d_1 \sin(\theta_1) - d_2 \sin(\theta_1 + \theta_2) & -d_2 \sin(\theta_1 + \theta_2) \\ d_1 \cos(\theta_1) + d_2 \cos(\theta_1 + \theta_2) & d_2 \cos(\theta_1 + \theta_2) \\ 1 & 1 \end{pmatrix}$$

Task space (m=3)

Remarks:

- $\boldsymbol{J}$ depends on the joint angles (θ_1, θ_2);
- $\boldsymbol{J}$ has as many columns as the number of joint angles (here: 2), and as many rows as the number of parameters of the end-effector (here: 3).

The Jacobian matrix is useful in that it gives the relationship between joint angle velocity $\dot{\boldsymbol{q}}$ and the end-effector velocity $\dot{\boldsymbol{p}}$:

$$\dot{\boldsymbol{p}} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}.$$

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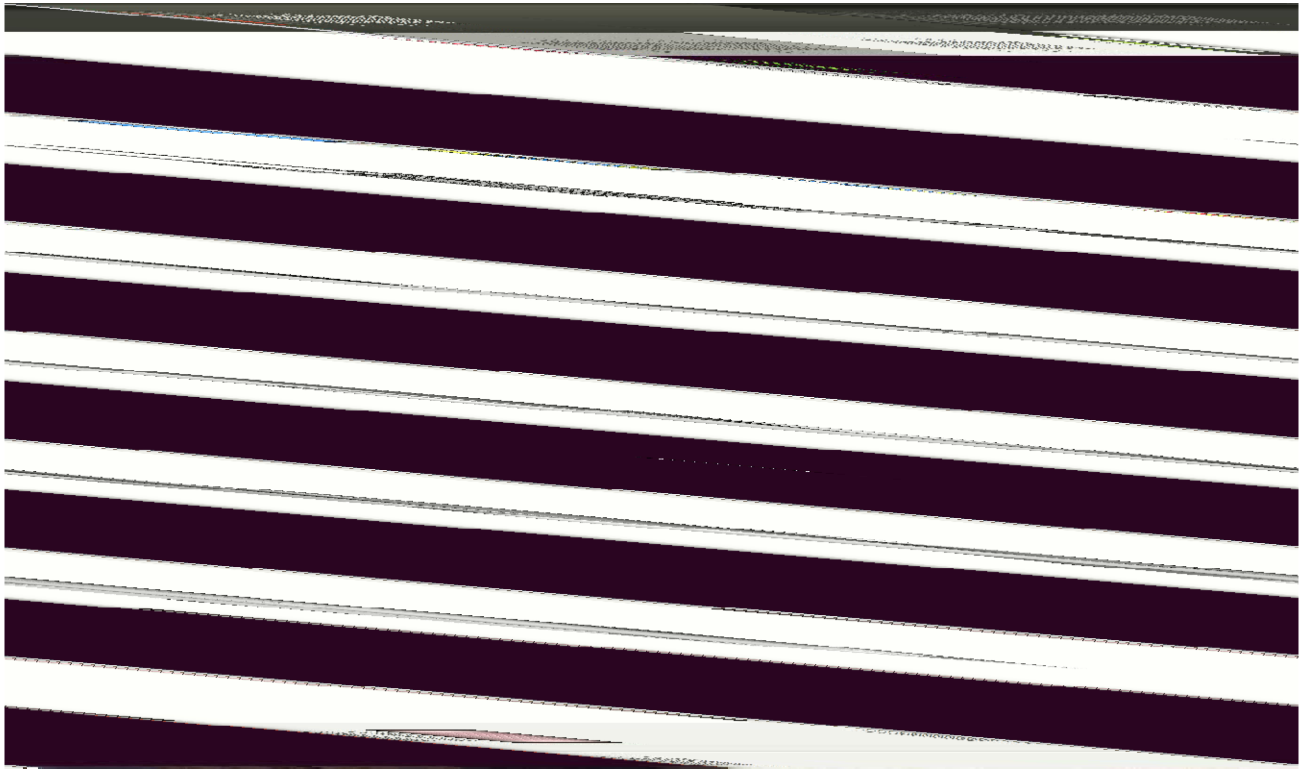
Simulation instruments

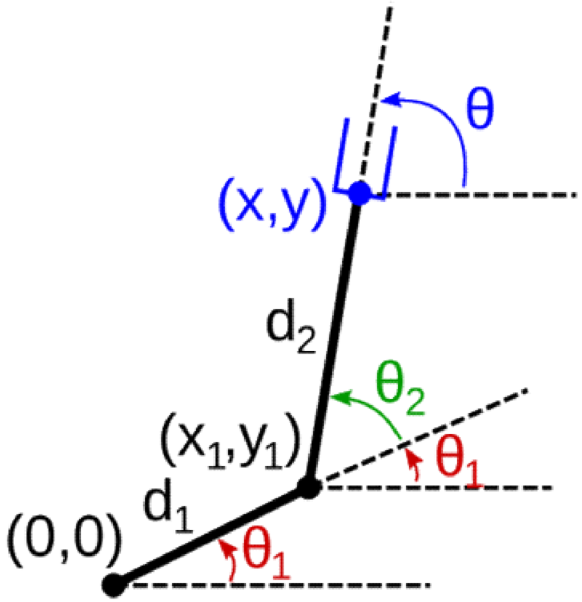
OpenRAVE

Open Robotics Automation Virtual Environment

http://openrave.org/



Example: inverse Kinematics



TASK:
to place the gripper at a desired position:

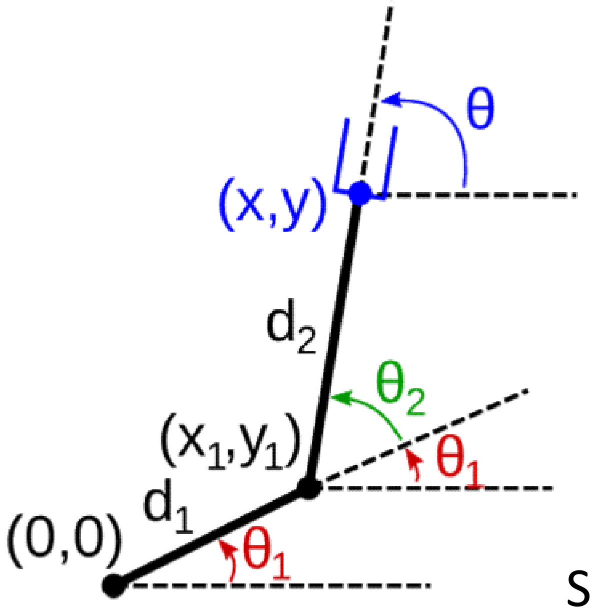
$$\boldsymbol{p}_{\text{des}} := (x_{\text{des}}, y_{\text{des}})$$

Finding the appropriate *joint angles* that achieve this position it constitutes the ***inverse kinematics*** problem:

$$\boldsymbol{q}^* := (\theta_1^*, \theta_2^*)$$

Unknown → (θ_1^*, θ_2^*)

Example: inverse Kinematics



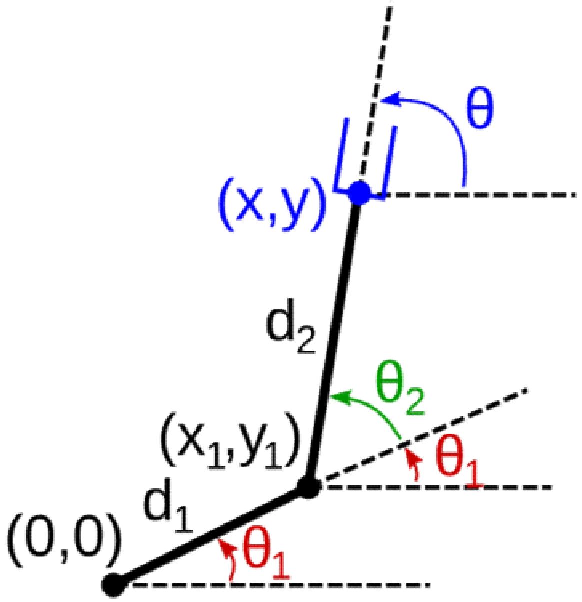
The forward kinematic provided:

$$\begin{cases} x_{\text{des}} &= d_1 \cos(\theta_1^*) + d_2 \cos(\theta_1^* + \theta_2^*) \\ y_{\text{des}} &= d_1 \sin(\theta_1^*) + d_2 \sin(\theta_1^* + \theta_2^*) \end{cases}$$

Squaring both sides of equation and summing them up:

$$\begin{aligned} x_{\text{des}}^2 + y_{\text{des}}^2 &= d_1^2 + d_2^2 + 2d_1 d_2 \left(\cos(\theta_1^*) \cos(\theta_1^* + \theta_2^*) + \sin(\theta_1^*) \sin(\theta_1^* + \theta_2^*)\right) \\ &= d_1^2 + d_2^2 + 2d_1 d_2 \cos(\theta_2^*). \end{aligned}$$

Example: inverse Kinematics



The forward kinematic provided:

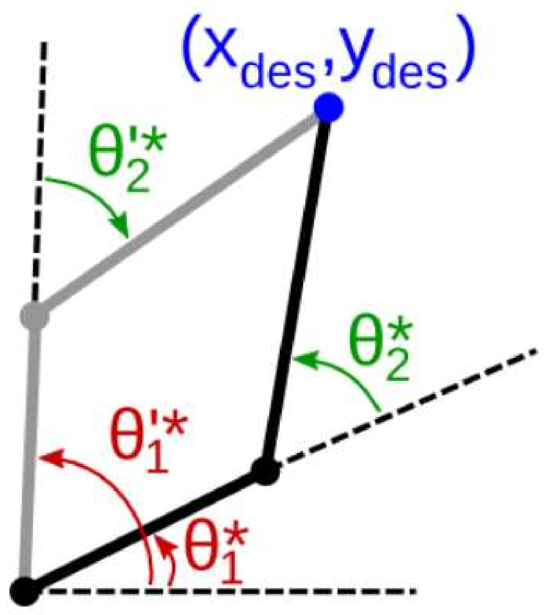
$$x_{\text{des}}^2 + y_{\text{des}}^2 \quad = \quad d_1^2 + d_2^2 + 2d_1 d_2 \cos(\theta_2^*).$$

$$\cos(\theta_2^*) = \frac{x_{\text{des}}^2 + y_{\text{des}}^2 - d_1^2 - d_2^2}{2d_1 d_2}.$$

$$\theta_2^* = \pm \arccos\left(\frac{x_{\text{des}}^2 + y_{\text{des}}^2 - d_1^2 - d_2^2}{2d_1 d_2}\right)$$

There are two values of the angle. Why?

Example: inverse Kinematics



Next, after some calculations, one can find the expression for the two angles:

$$\theta_2^* = \pm \arccos\left(\frac{x_{\text{des}}^2 + y_{\text{des}}^2 - d_1^2 - d_2^2}{2d_1 d_2}\right)$$

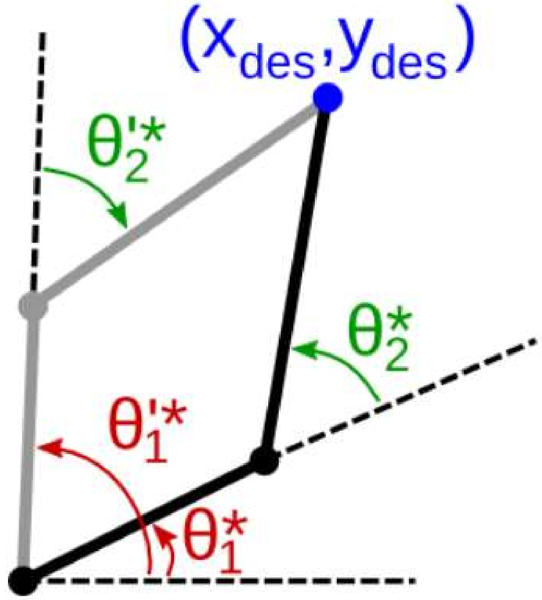
$$\theta_1^* = \arctan 2(y_{\text{des}}, x_{\text{des}}) - \arctan 2(k_2, k_1),$$

where

$$k_1 := d_1 + d_2 \cos(\theta_2^*) \quad \text{and} \quad k_2 := d_2 \sin(\theta_2^*).$$

Appendix for compete calculation.

EXAMPLE: INVERSE KINEMATICS



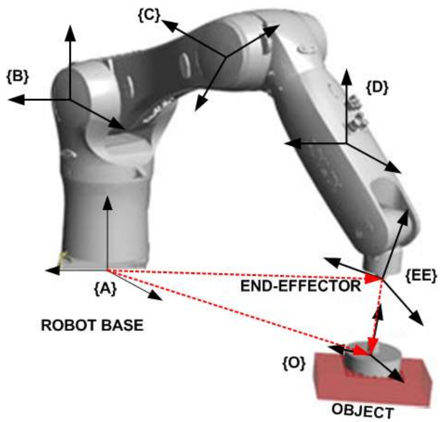
The above derivations raise the following remarks:

- Inverse kinematics calculations are in general much more difficult than forward kinematics calculations;
- While a configuration $\boldsymbol{q}$ always yields *one* forward kinematics solution $\boldsymbol{p}$, a given desired end-effector position $\boldsymbol{p}_{\text{des}}$ may correspond to zero, one, or multiple possible IK solutions $\boldsymbol{q}^*$.

Geometry recall

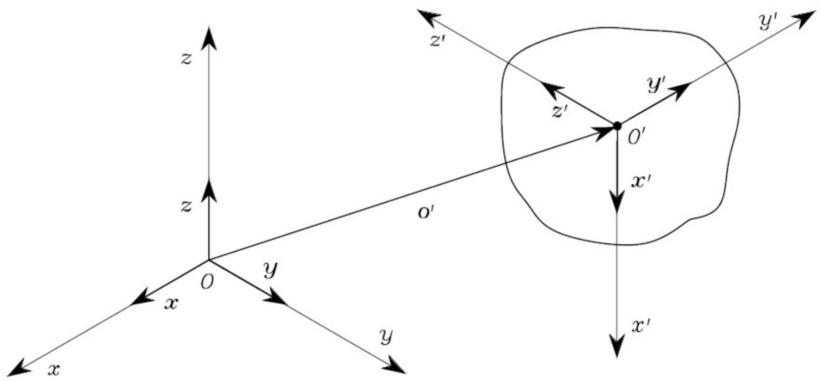
In order to represent the relative position and orientation of one rigid body with respect to another, we will rigidly attach coordinate frames to each body, and then specify the geometric relationships between these coordinate frames.

We begin with the case of rotations in the plane, and then generalize our results to the case of orientations in a three dimensional space.



Pose of a Rigid Body

A *rigid body* is completely described in space by its *position* and *orientation* with respect to a reference frame.



O–xyz be the orthonormal reference frame and ***x***, ***y***, ***z*** be the unit vectors of the frame axes.

The position of a point ***O′*** on the rigid body with respect to the coordinate frame *O–xyz* is expressed by the relation:

$$\mathbf{o}' = o'_x \boldsymbol{x} + o'_y \boldsymbol{y} + o'_z \boldsymbol{z},$$

In order to describe the rigid body orientation, it is convenient to consider an orthonormal frame attached to the body and express its **unit vectors** with respect to the reference frame:

$$\boldsymbol{x}' = x'_x \boldsymbol{x} + x'_y \boldsymbol{y} + x'_z \boldsymbol{z}$$
$$\boldsymbol{y}' = y'_x \boldsymbol{x} + y'_y \boldsymbol{y} + y'_z \boldsymbol{z}$$
$$\boldsymbol{z}' = z'_x \boldsymbol{x} + z'_y \boldsymbol{y} + z'_z \boldsymbol{z}.$$

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Rotation Matrix

the three unit vectors are describing the body orientation with respect to the reference frame can be combined in the (3 × 3) matrix

$$\boldsymbol{R} = \begin{bmatrix} \boldsymbol{x}' & \boldsymbol{y}' & \boldsymbol{z}' \end{bmatrix} = \begin{bmatrix} x'_x & y'_x & z'_x \\ x'_y & y'_y & z'_y \\ x'_z & y'_z & z'_z \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}'^T\boldsymbol{x} & \boldsymbol{y}'^T\boldsymbol{x} & \boldsymbol{z}'^T\boldsymbol{x} \\ \boldsymbol{x}'^T\boldsymbol{y} & \boldsymbol{y}'^T\boldsymbol{y} & \boldsymbol{z}'^T\boldsymbol{y} \\ \boldsymbol{x}'^T\boldsymbol{z} & \boldsymbol{y}'^T\boldsymbol{z} & \boldsymbol{z}'^T\boldsymbol{z} \end{bmatrix}$$

which is termed ***rotation matrix***

It is worth noting that the column vectors of matrix ***R*** are mutually orthogonal since they represent the unit vectors of an orthonormal frame,

$$\boldsymbol{x}'^T\boldsymbol{y}' = 0 \qquad \boldsymbol{y}'^T\boldsymbol{z}' = 0 \qquad \boldsymbol{z}'^T\boldsymbol{x}' = 0.$$

R is an *orthogonal* matrix meaning that $\boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{I}_3$

Rotation Matrix

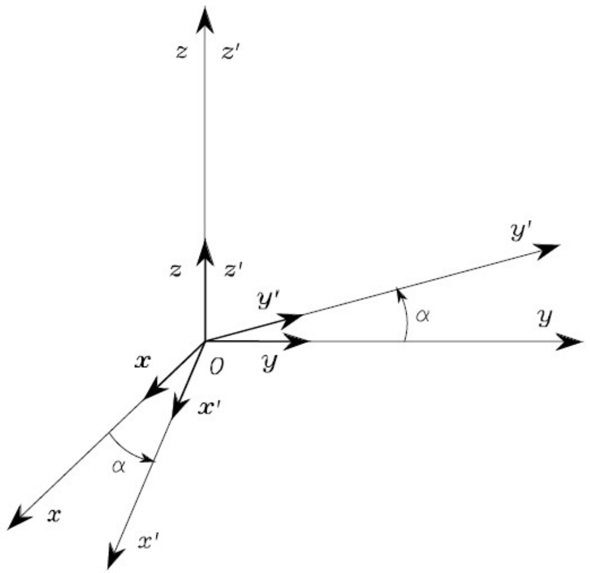
If both sides of $\boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{I}_3$ are postmultiplied by the inverse matrix $\boldsymbol{R}^{-1}$: $\boldsymbol{R}^T = \boldsymbol{R}^{-1}$

The above-defined rotation matrix belongs to the *special orthonormal group* $SO(m)$ of the real $(m \times m)$ matrices with othonormal columns and determinant equal to 1; in the case of spatial rotations it is $m = 3$, whereas in the case of planar rotations it is $m = 2$.

Elementary Rotations

Suppose that the reference frame *O–xyz* is rotated by an angle *α* about axis *z,* and let O'–$x'y'z'$ be the rotated frame.

The unit vectors of the new frame can be described in terms of their components with respect to the reference frame *O–xyz*.



$$\boldsymbol{x}' = \begin{bmatrix} \cos\alpha \\ \sin\alpha \\ 0 \end{bmatrix} \qquad \boldsymbol{y}' = \begin{bmatrix} -\sin\alpha \\ \cos\alpha \\ 0 \end{bmatrix} \qquad \boldsymbol{z}' = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Elementary Rotations

Hence, the rotation matrix of frame $O'–x'y'z'$ with respect to frame $O–xyz$ by an angle α about axis z is:

$$\boldsymbol{R}_z(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In a similar manner if the rotation are cording other axes we obtain:

$$\boldsymbol{R}_y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \qquad \boldsymbol{R}_x(\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix}$$

It is easy to verify that $\boldsymbol{R}_k(-\vartheta) = \boldsymbol{R}_k^T(\vartheta) \quad k = x, y, z.$

Representation of a Vector

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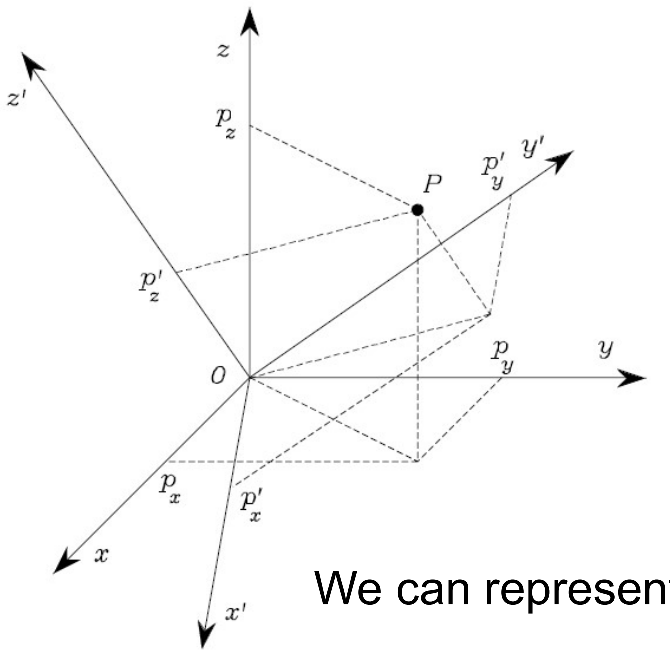
Consider the case when the origin of the body frame coincides with the origin of the reference frame



A point *P* in space can be represented in the two frames

$$\boldsymbol{p} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \quad \text{with respect to frame } O\text{–}xyz$$

$$\boldsymbol{p}' = \begin{bmatrix} p'_x \\ p'_y \\ p'_z \end{bmatrix} \quad \text{with respect to frame } O\text{–}x'y'z'$$

We can represent P in terms of the reference O-xyz:

$$\boldsymbol{p} = p'_x \boldsymbol{x}' + p'_y \boldsymbol{y}' + p'_z \boldsymbol{z}' = \begin{bmatrix} \boldsymbol{x}' & \boldsymbol{y}' & \boldsymbol{z}' \end{bmatrix} \boldsymbol{p}'$$

$$\boldsymbol{p} = \boldsymbol{R}\boldsymbol{p}'.$$

Representation of a Vector

The rotation matrix ***R*** represents the *transformation matrix* of the vector coordinates in frame *O–xyz* into the coordinates of the same vector in frame $O'–x'y'z'$.

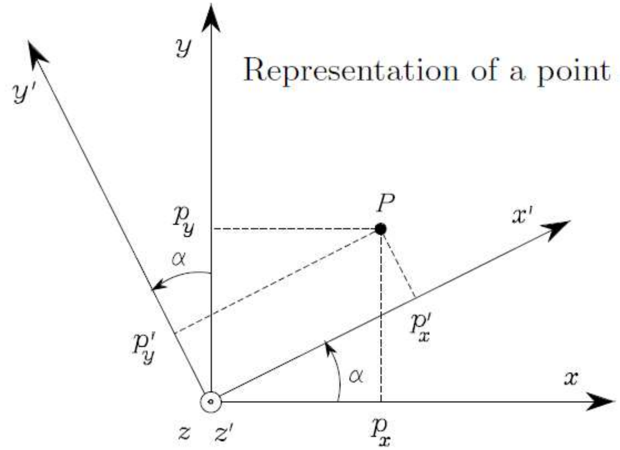
$$\boldsymbol{p} = \boldsymbol{R}\boldsymbol{p}'.$$

In view of the orthogonality property, the inverse transformation is simply given by

$$\boldsymbol{R}_k(-\vartheta) = \boldsymbol{R}_k^T(\vartheta) \quad k = x, y, z.$$

$$\boldsymbol{p}' = \boldsymbol{R}^T\boldsymbol{p}.$$

Elementary Rotations: Example



Consider two frames with common origin mutually rotated by an angle α about the axis z. Let $\boldsymbol{p}$ and $\boldsymbol{p}'$ be the vectors of the coordinates of a point P, expressed in the frames O–xyz and O–$x'y'z'$, respectively.

the relationship between the coordinates of P in the two frames is

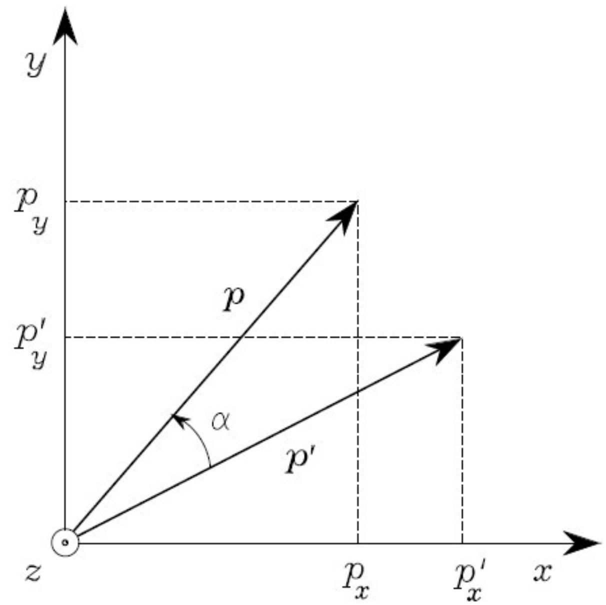
$$
\begin{aligned}
p_x &= p'_x \cos\alpha - p'_y \sin\alpha \\
p_y &= p'_x \sin\alpha + p'_y \cos\alpha \\
p_z &= p'_z.
\end{aligned}
$$

Rotation of a Vector

Consider the vector $\boldsymbol{p}$ which is obtained by rotating a vector $\boldsymbol{p}'$ in the plane xy by an angle α about axis z of the reference frame .

Let (p'_x, p'_y, p'_z) be the coordinates of the vector $\boldsymbol{p}'$. The vector $\boldsymbol{p}$ has components

$$
\begin{aligned}
p_x &= p'_x \cos\alpha - p'_y \sin\alpha \\
p_y &= p'_x \sin\alpha + p'_y \cos\alpha \\
p_z &= p'_z .
\end{aligned}
$$



It is easy to recognize that $\boldsymbol{p}$ can be expressed as $\boldsymbol{p} = \boldsymbol{R}_z(\alpha)\boldsymbol{p}'$,

where $\boldsymbol{R}_z(\alpha)$ is the same rotation matrix as in

$$
\boldsymbol{R}_z(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

Summary of Rotation Matrix

In sum, a rotation matrix attains three *equivalent geometrical meanings*:

- It describes the mutual orientation between two coordinate frames; its column vectors are the direction cosines of the axes of the rotated frame with respect to the original frame.
- It represents the coordinate transformation between the coordinates of a point expressed in two different frames (with common origin).
- It is the operator that allows the rotation of a vector in the same coordinate frame.

Composition of Rotation Matrices

to derive composition rules of rotation matrices, it is useful to consider the expression of a vector in two different reference frames

Let then $O–x_0y_0z_0$, $O–x_1y_1z_1$, $O–x_2y_2z_2$ be three frames with common origin O

The vector $\boldsymbol{p}$ describing the position of a generic point in space can be expressed in each of the above frames; let $\boldsymbol{p}^0$, $\boldsymbol{p}^1$, $\boldsymbol{p}^2$

R^j_i denotes the rotation matrix of Frame i with respect to Frame j

$$\boldsymbol{p}^1 = \boldsymbol{R}^1_2 \boldsymbol{p}^2.$$

$$\boldsymbol{p}^0 = \boldsymbol{R}^0_1 \boldsymbol{p}^1$$

$$\boldsymbol{p}^0 = \boldsymbol{R}^0_2 \boldsymbol{p}^2.$$

Property: Multiplication of rotation matrix provides different orientation for composite rotation

$$\boldsymbol{R}^0_2 = \boldsymbol{R}^0_1 \boldsymbol{R}^1_2.$$

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Composition of Rotation Matrices
in the call you may find different notation

R^j_i denotes the rotation matrix **from Frame *i*** with respect **to Frame *j***

R^{to}_{from} $\quad$ ${}^{to}R_{from}$ $\quad$ ${}^{to}_{from}R$

You may find in the class these three notation with superscript and subscript on LEFT or RIGHT but the meaning is the same!
The meaning is that:
<u>What is in the bottom is FROM what reference we want to start</u>
<u>What is on top is TO what reference frame we want to go</u>

Euler Angles

A representation of orientation in terms of three independent parameters constitutes a ***minimal representation***

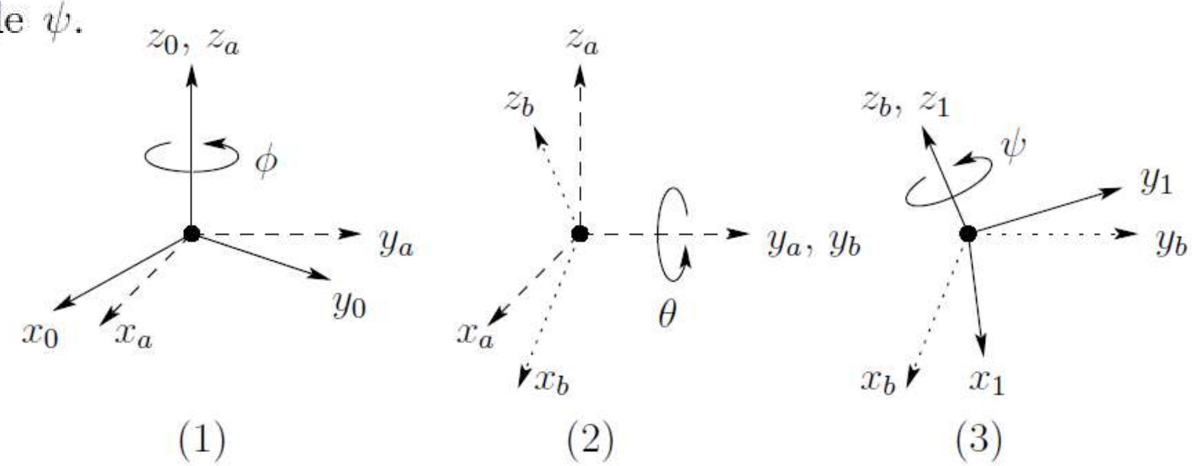
A minimal representation of orientation can be obtained by using a set of three angles

$$\boldsymbol{\phi} = [\varphi \quad \vartheta \quad \psi]^T$$

First rotate about the z-axis by the angle ϕ.

Next rotate about the current y-axis by the angle θ.

Finally rotate about the current z-axis by the angle ψ.



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Euler ZYZ Angles

The rotation described by *ZYZ angles* is obtained as composition of the following elementary rotations :

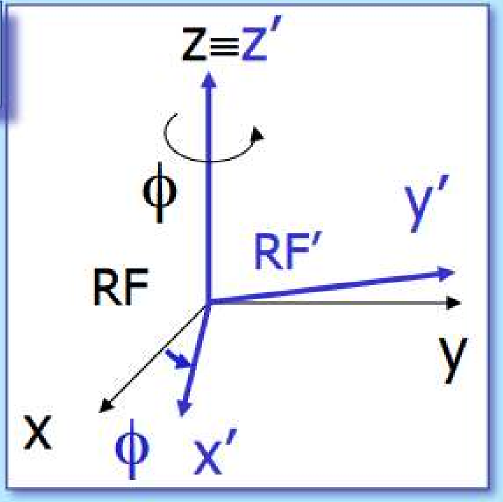
- Rotate the reference frame by the angle φ about axis z; this rotation is described by the matrix $\boldsymbol{R}_z(\varphi)$
- Rotate the current frame by the angle ϑ about axis y'; this rotation is described by the matrix $\boldsymbol{R}_{y'}(\vartheta)$
- Rotate the current frame by the angle ψ about axis z''; this rotation is described by the matrix $\boldsymbol{R}_{z''}(\psi)$

The resulting frame orientation is obtained by composition of rotations with respect to *current frames*,

$$\begin{aligned}\boldsymbol{R}(\boldsymbol{\phi}) &= \boldsymbol{R}_z(\varphi)\boldsymbol{R}_{y'}(\vartheta)\boldsymbol{R}_{z''}(\psi)\\ &= \begin{bmatrix} c_\varphi c_\vartheta c_\psi - s_\varphi s_\psi & -c_\varphi c_\vartheta s_\psi - s_\varphi c_\psi & c_\varphi s_\vartheta \\ s_\varphi c_\vartheta c_\psi + c_\varphi s_\psi & -s_\varphi c_\vartheta s_\psi + c_\varphi c_\psi & s_\varphi s_\vartheta \\ -s_\vartheta c_\psi & s_\vartheta s_\psi & c_\vartheta \end{bmatrix}\end{aligned}$$

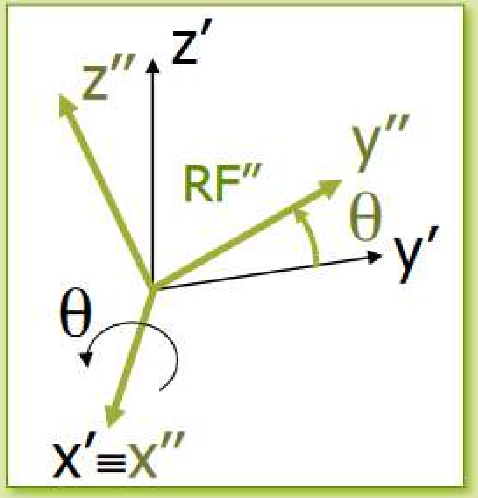
ZX′Z″ Euler angles

1



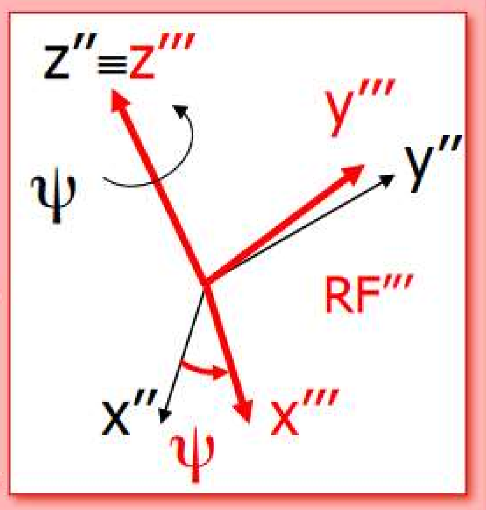
$$R_z(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2



$$R_{x'}(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

3



$$R_{z''}(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Inverse problem

$$\boldsymbol{R}(\phi) = \boldsymbol{R}_z(\varphi)\boldsymbol{R}_{y'}(\vartheta)\boldsymbol{R}_{z''}(\psi) = \begin{bmatrix} c_\varphi c_\vartheta c_\psi - s_\varphi s_\psi & -c_\varphi c_\vartheta s_\psi - s_\varphi c_\psi & c_\varphi s_\vartheta \\ s_\varphi c_\vartheta c_\psi + c_\varphi s_\psi & -s_\varphi c_\vartheta s_\psi + c_\varphi c_\psi & s_\varphi s_\vartheta \\ -s_\vartheta c_\psi & s_\vartheta s_\psi & c_\vartheta \end{bmatrix}$$

It is useful to solve the *inverse problem*, that is to determine the **set of Euler** angles corresponding to a given rotation matrix (known)

$$\boldsymbol{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

By considering the elements [1, 3] and [2, 3] $\quad \varphi = \text{Atan2}(r_{23}, r_{13})$

The function $\text{Atan2}(y, x)$ computes the arctangent of the ratio y/x but utilizes the sign of each argument to determine which quadrant the resulting angle belongs to; this allows the correct determination of an angle in a range of 2π.

Inverse problem

Then, squaring and summing the elements [1, 3] and [2, 3] and using the element [3, 3] yields

$$\vartheta = \mathrm{Atan2}\left(\sqrt{r_{13}^2 + r_{23}^2}, r_{33}\right)$$

The choice of the positive sign for the term $r^2_{13} + r^2_{23}$ limits the range of feasible values of ϑ to $(0, \pi)$.

On this assumption, considering the elements [3, 1] and [3, 2] gives

$$\psi = \mathrm{Atan2}(r_{32}, -r_{31})$$

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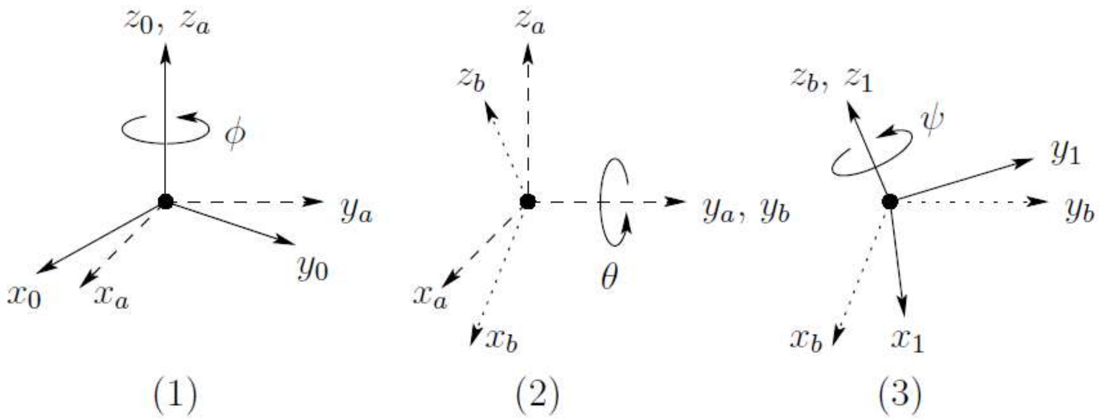
Summary



Direct problem

Knowing the angles → find the matrix

$$\boldsymbol{R}(\phi) = \boldsymbol{R}_z(\varphi)\boldsymbol{R}_{y'}(\vartheta)\boldsymbol{R}_{z''}(\psi) = \begin{bmatrix} c_\varphi c_\vartheta c_\psi - s_\varphi s_\psi & -c_\varphi c_\vartheta s_\psi - s_\varphi c_\psi & c_\varphi s_\vartheta \\ s_\varphi c_\vartheta c_\psi + c_\varphi s_\psi & -s_\varphi c_\vartheta s_\psi + c_\varphi c_\psi & s_\varphi s_\vartheta \\ -s_\vartheta c_\psi & s_\vartheta s_\psi & c_\vartheta \end{bmatrix}$$

Inverse problem

Knowing the matrix → find the angles

$$\boldsymbol{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$\varphi = \text{Atan2}(r_{23}, r_{13})$$

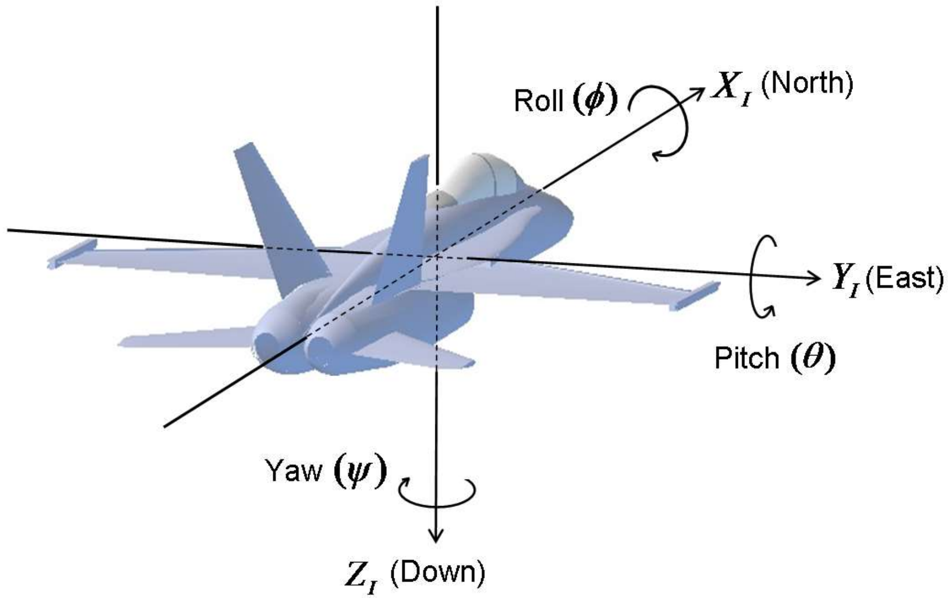
$$\vartheta = \text{Atan2}\left(\sqrt{r_{13}^2 + r_{23}^2}, r_{33}\right)$$

$$\psi = \text{Atan2}(r_{32}, -r_{31}).$$

RPY Angles

Another set of Euler angles originates from a representation of orientation in the (aero)nautical field.

These are the ZYX angles, also called *Roll–Pitch–Yaw angles*, to denote the typical changes of attitude of an (air)craft.

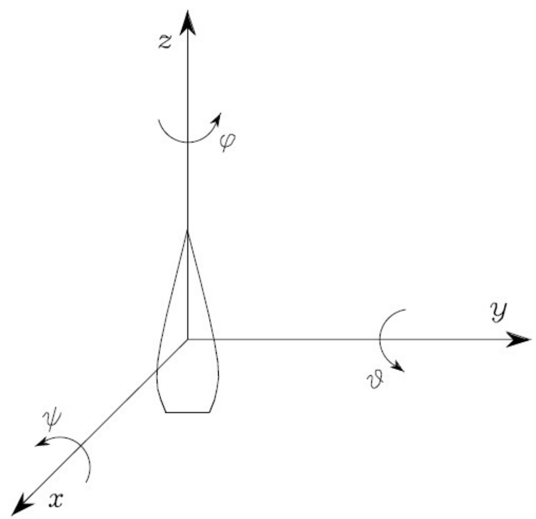


In this case, the angles $\boldsymbol{\varphi} = [\phi \;\; \vartheta \;\; \psi]^T$ represent rotations defined with respect to a fixed frame attached to the centre of mass of the craft

RPY Angles

The rotation resulting from Roll–Pitch–Yaw angles can be obtained as follows:

- Rotate the reference frame by the angle ψ about axis x (yaw); this rotation is described by the matrix $\boldsymbol{R}_x(\psi)$
- Rotate the reference frame by the angle ϑ about axis y (pitch); this rotation is described by the matrix $\boldsymbol{R}_y(\vartheta)$
- Rotate the reference frame by the angle φ about axis z (roll); this rotation is described by the matrix $\boldsymbol{R}_z(\varphi)$



RPY Angles

The resulting frame orientation (*direct solution*) is obtained by composition of rotations with respect to the ***fixed frame***

$$\boldsymbol{R}(\phi) = \boldsymbol{R}_z(\varphi)\boldsymbol{R}_y(\vartheta)\boldsymbol{R}_x(\psi) = \begin{bmatrix} c_\varphi c_\vartheta & c_\varphi s_\vartheta s_\psi - s_\varphi c_\psi & c_\varphi s_\vartheta c_\psi + s_\varphi s_\psi \\ s_\varphi c_\vartheta & s_\varphi s_\vartheta s_\psi + c_\varphi c_\psi & s_\varphi s_\vartheta c_\psi - c_\varphi s_\psi \\ -s_\vartheta & c_\vartheta s_\psi & c_\vartheta c_\psi \end{bmatrix}$$

As for the Euler angles ZYZ, the *inverse solution* to a given rotation matrix

$$\boldsymbol{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

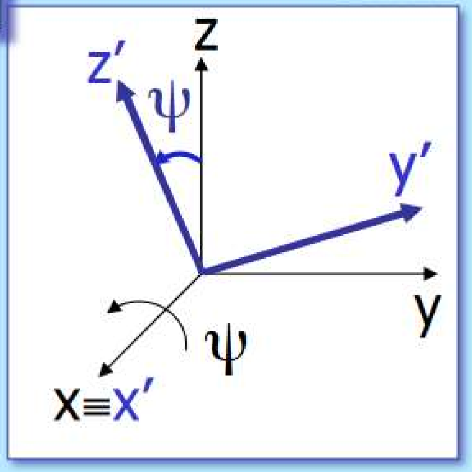
$$\varphi = \mathrm{Atan2}(r_{21}, r_{11})$$

$$\vartheta = \mathrm{Atan2}\left(-r_{31}, \sqrt{r_{32}^2 + r_{33}^2}\right)$$

$$\psi = \mathrm{Atan2}(r_{32}, r_{33}).$$

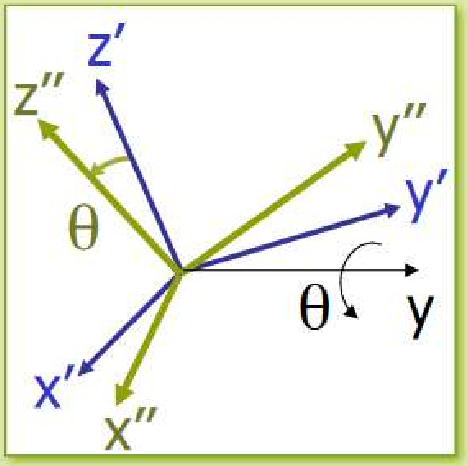
Roll-Pitch-Yaw angles

1 **ROLL**



$$R_X(\psi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{bmatrix}$$

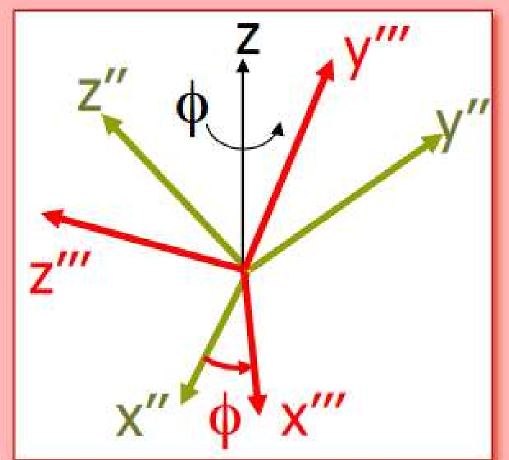
PITCH **2**



$C_1 R_Y(\theta) C_1^T$

with $R_Y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$

3 **YAW**



$C_2 R_Z(\phi) C_2^T$

with $R_Z(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$