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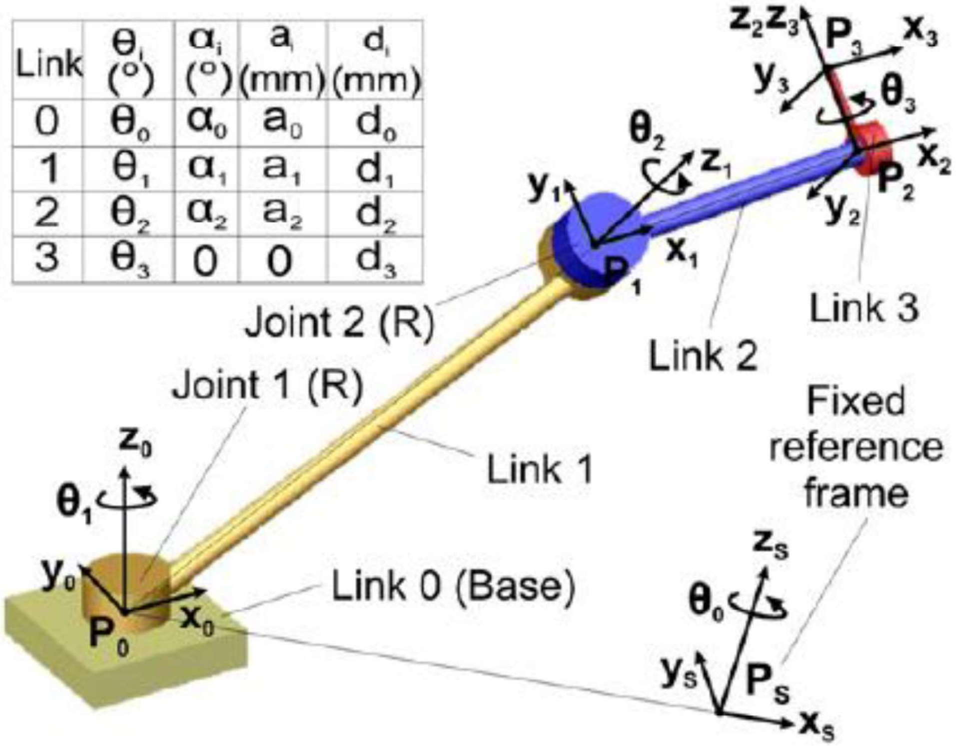
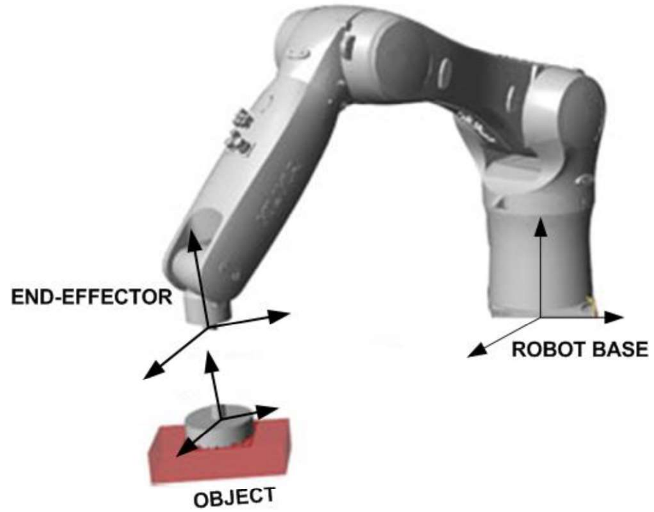
Robotics 1
Homogeneous Transformation
Denavit & Hartenberg Notation



Link	θ_i (°)	α_i (°)	a_i (mm)	d_i (mm)
0	θ_0	α_0	a_0	d_0
1	θ_1	α_1	a_1	d_1
2	θ_2	α_2	a_2	d_2
3	θ_3	0	0	d_3

Kinematics: spatial description and transformation

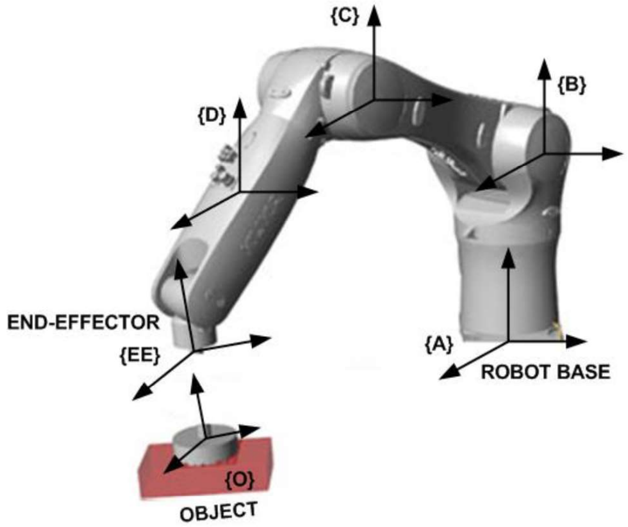
Robotic manipulation implies multiple actions:
- Moving tools
- Picking objects
- Assembling parts

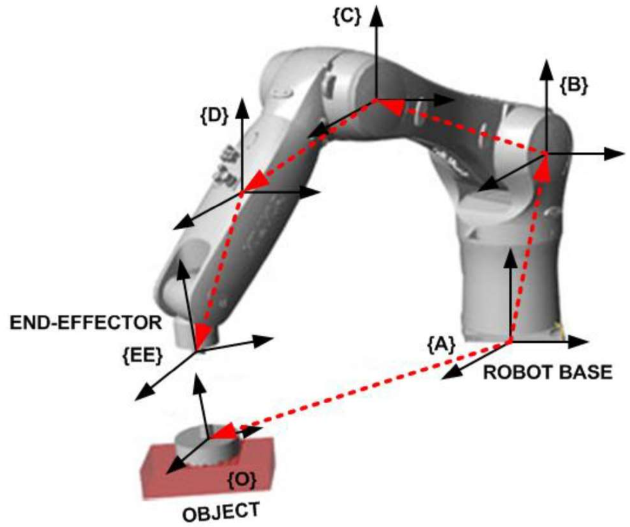


We must relate the kinematics of the **object** to be manipulated with the one of the **robotic manipulator**.

Both the robot and the object must have Reference Frames

KINEMATICS: SPATIAL DESCRIPTION AND TRANSFORMATION





Coordinated Reference frames
- the robot
- the object

Finding the Transformations among the Reference frames

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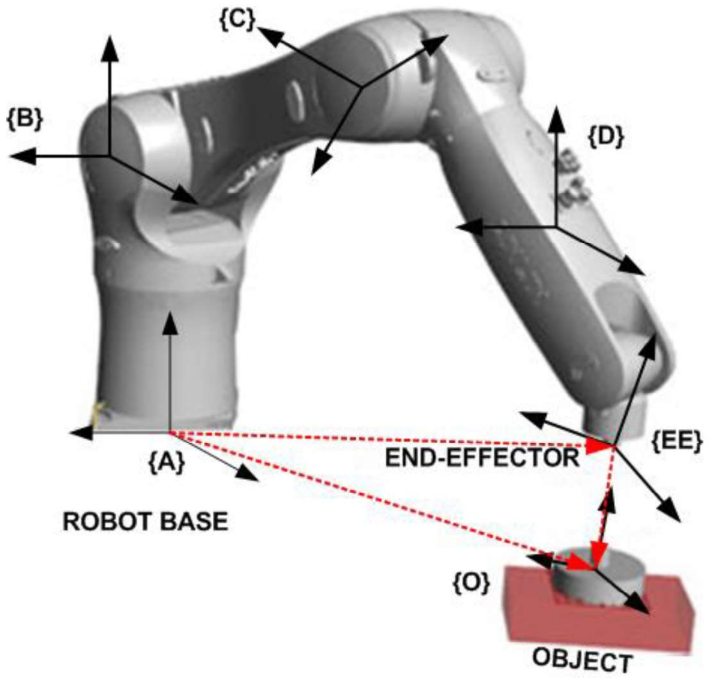
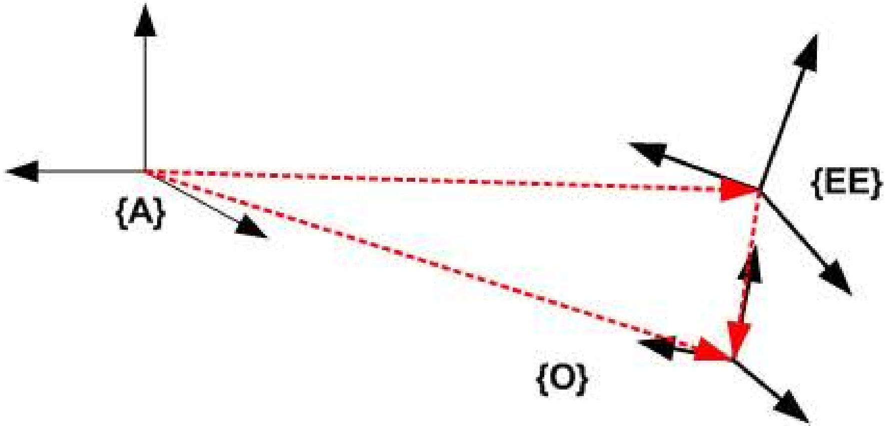
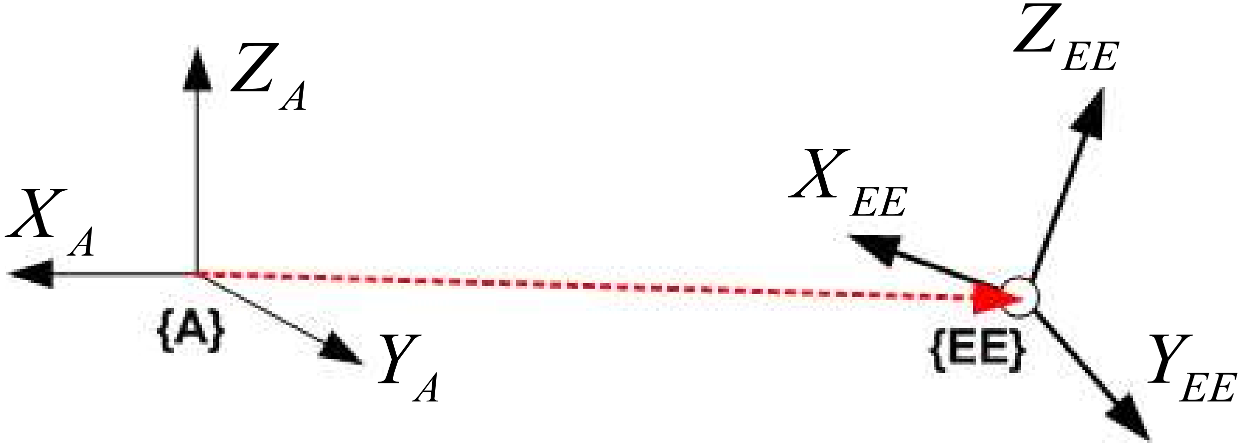






Writing the orthonormal base of the reference frame **{EE}** in terms of the coordinates of the the **{A}** frame, we obtain the following <u>Rotation Matrix</u>:

$$ {}^{A}_{EE}R = [{}^{A}X_{EE}, {}^{A}Y_{EE}, {}^{A}Z_{EE}] = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} $$

The set of the three vectors specifes the orientation of **{EE}** respect to **{A}**.

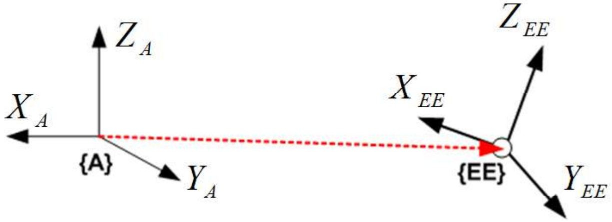
We can express also the components r_{ij} as product of *unit vectors* of the orthonormal bases of **{A}** and **{EE}.**

$$ {}^{A}_{EE}R = [{}^{A}X_{EE}, {}^{A}Y_{EE}, {}^{A}Z_{EE}] = \begin{bmatrix} X_{EE}\cdot X_A & Y_{EE}\cdot X_A & Z_{EE}\cdot X_A \\ X_{EE}\cdot Y_A & Y_{EE}\cdot Y_A & Z_{EE}\cdot Y_A \\ X_{EE}\cdot Z_A & Y_{EE}\cdot Z_A & Z_{EE}\cdot Z_A \end{bmatrix} $$

X_{EE} expressed in {A} ${}^{A}X_{EE}$

Y_{EE} expressed in {A} ${}^{A}Y_{EE}$

Z_{EE} expressed in {A} ${}^{A}Z_{EE}$



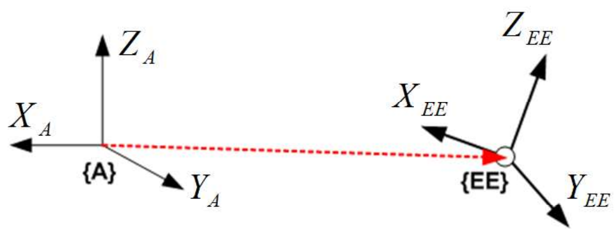
We can express also the components r_{ij} as product of *unit vectors* of the orthonormal bases of {A} and {EE}.

$$
{}^{A}_{EE}R = [{}^{A}X_{EE}, {}^{A}Y_{EE}, {}^{A}Z_{EE}] = \begin{bmatrix} X_{EE}\cdot X_{A} & Y_{EE}\cdot X_{A} & Z_{EE}\cdot X_{A} \\ X_{EE}\cdot Y_{A} & Y_{EE}\cdot Y_{A} & Z_{EE}\cdot Y_{A} \\ X_{EE}\cdot Z_{A} & Y_{EE}\cdot Z_{A} & Z_{EE}\cdot Z_{A} \end{bmatrix}
$$

X_A expressed in {EE} ${}^{EE}X_A$

Y_A expressed in {EE} ${}^{EE}Y_A$

Z_A expressed in {EE} ${}^{EE}Z_A$



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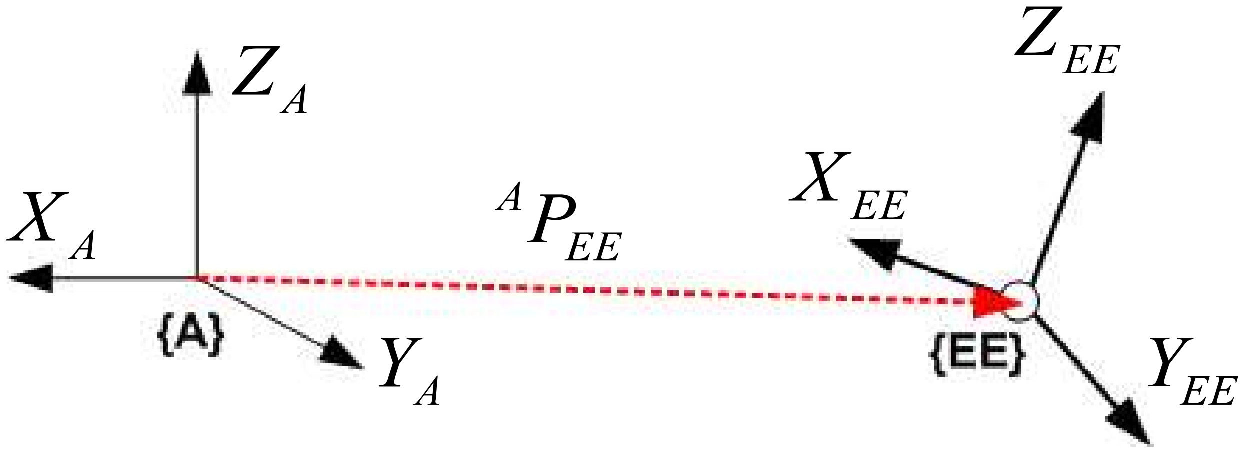
Property of the rotation matrix

$$ {}^{A}_{EE}R = [{}^{A}X_{EE}, {}^{A}Y_{EE}, {}^{A}Z_{EE}] = \begin{bmatrix} {}^{EE}X_{A}^{T} \\ {}^{EE}Y_{A}^{T} \\ {}^{EE}Z_{A}^{T} \end{bmatrix} = {}^{EE}_{A}R^{T} $$

Hence is possible to express the rotation of the frame **{A}** respect to the frame **{EE}** using the transpose of the matrix:

$$ {}^{EE}_{A}R = {}^{A}_{EE}R^{T} $$



The frame **{EE}** is **not only *rotated* respect** to **{A}** but ***also translated*.**
In robotics a **Frame** is an entity described by 4 vectors:

•3 vectors defining the rotation matrix→ ${}^{A}_{EE}R$

•1 vector defining the translation→ ${}^{A}P_{EE}$

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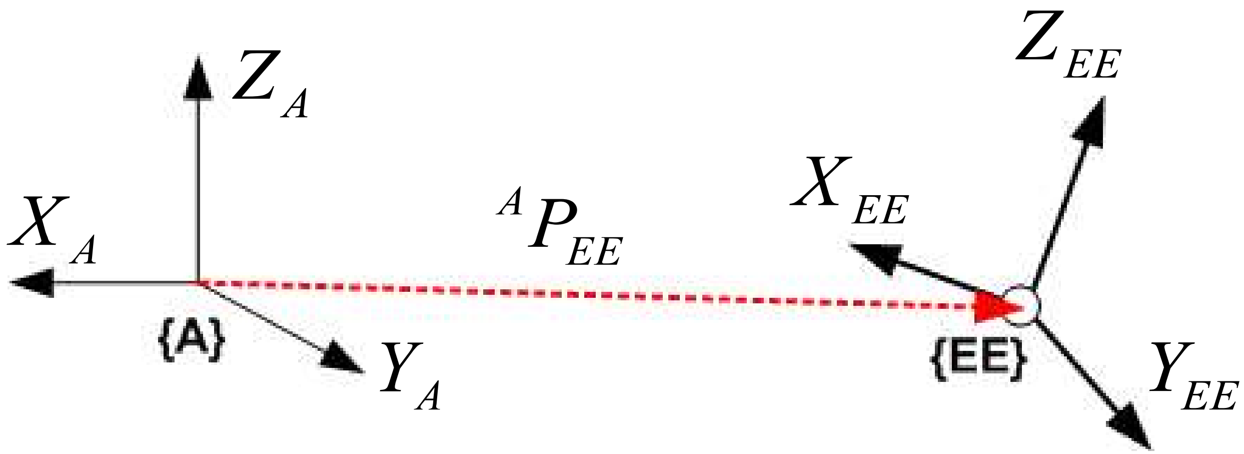
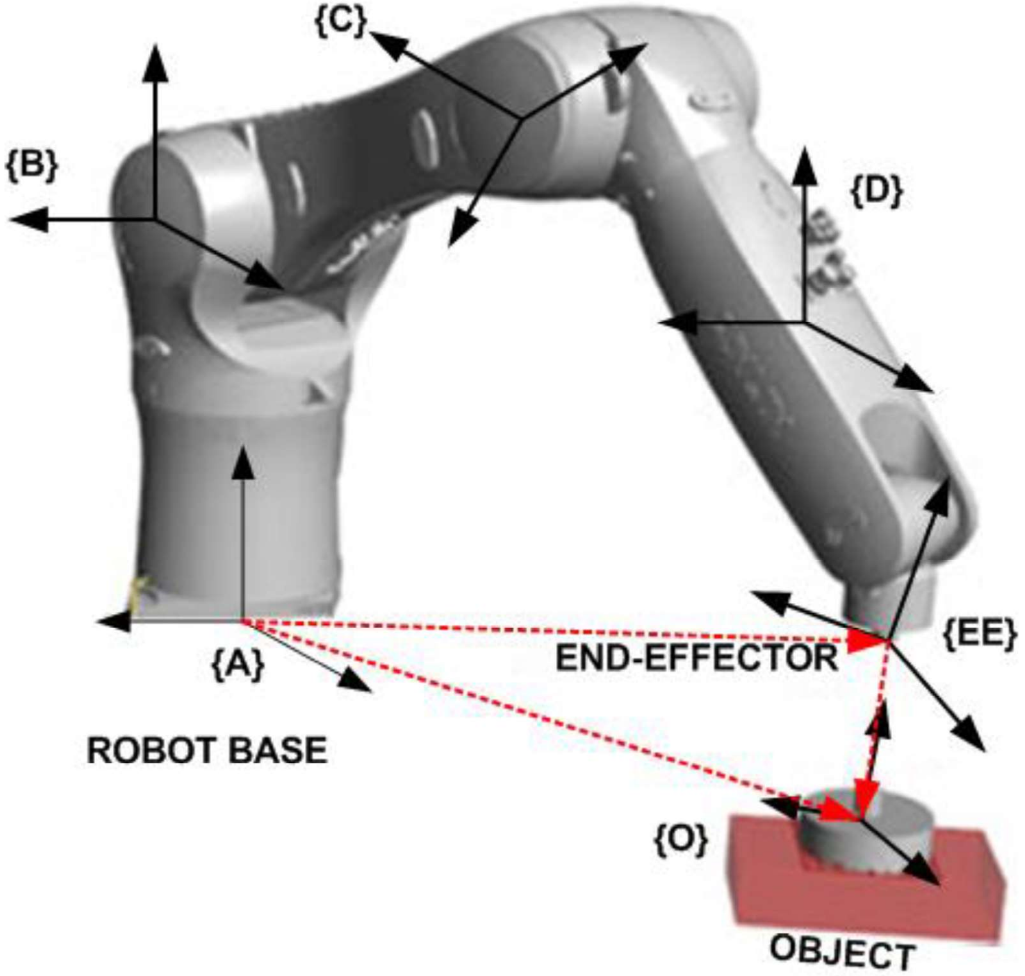
Evaluating the translation

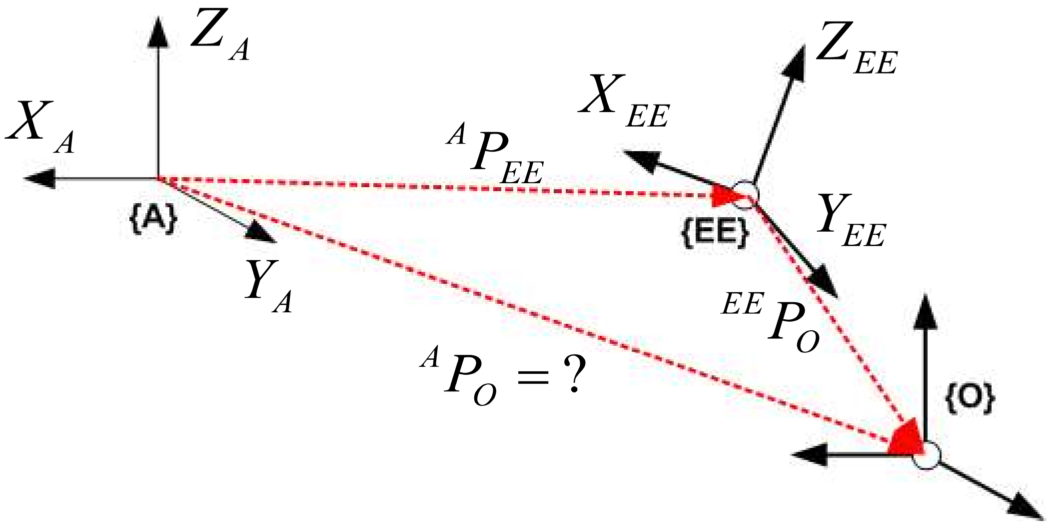


$^{A}P_{EE}$ is the *projection of the position* of **{EE}** into the unit vectors of the frame **{A}**.

$$^{A}P_{EE} = \begin{bmatrix} P_{EE} \cdot X_A \\ P_{EE} \cdot Y_A \\ P_{EE} \cdot Z_A \end{bmatrix}$$

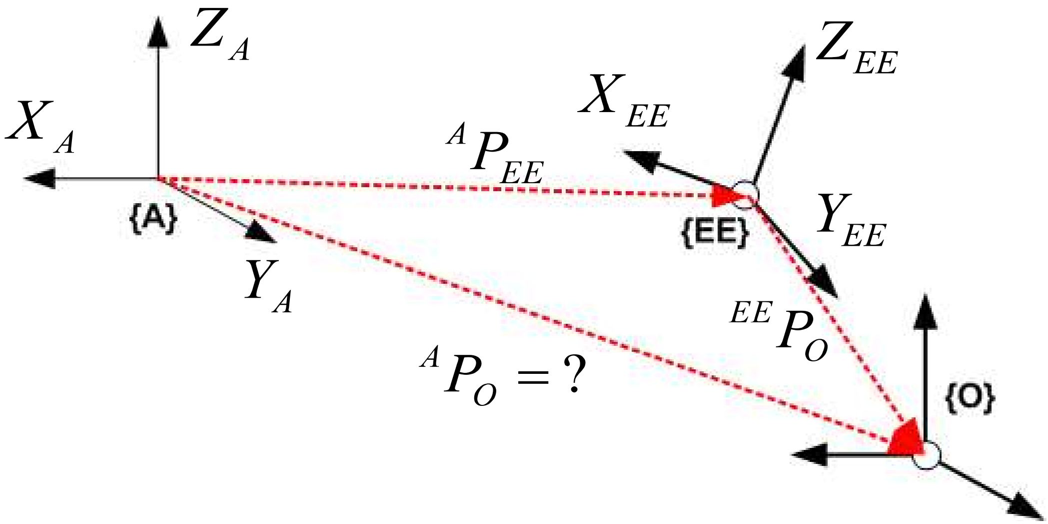
Base + end-effector + object





General case when we know the position of an object ***{O}*** *in a frame* ***{EE}*** *and we want to know its position respect to the frame* ***{A}***

- ***{EE}*** *is translated respect to* ***{A}*** → $^{A}P_{EE}$
- ***{EE}*** *is rotated respect to* ***{A}*** → $^{A}_{EE}R$
- ***{O}*** *is represented in* ***{EE}*** *by* → $^{EE}P_O$
- ***{O}*** *must be known also respect to* ***{A}*** → $^{A}P_O$



How to evaluate $^{A}P_O$?

$^{EE}P_O$ from **{EE}** must be expressed in a frame of the same orientation of **{A}**

We account of the translation between the origin of **{EE}** and **{A}**

$$^{A}P_O = {}^{A}_{EE}R \cdot {}^{EE}P_O + {}^{A}P_{EE}$$

Orientation (${}^{A}_{EE}R \cdot {}^{EE}P_O$)

Translation (${}^{A}P_{EE}$)

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$$ {}^{A}P_{O} = {}^{A}_{EE}R \cdot {}^{EE}P_{O} + {}^{A}P_{EE} $$

$$ {}^{A}P_{O} = {}^{A}_{EE}T \cdot {}^{EE}P_{O} $$

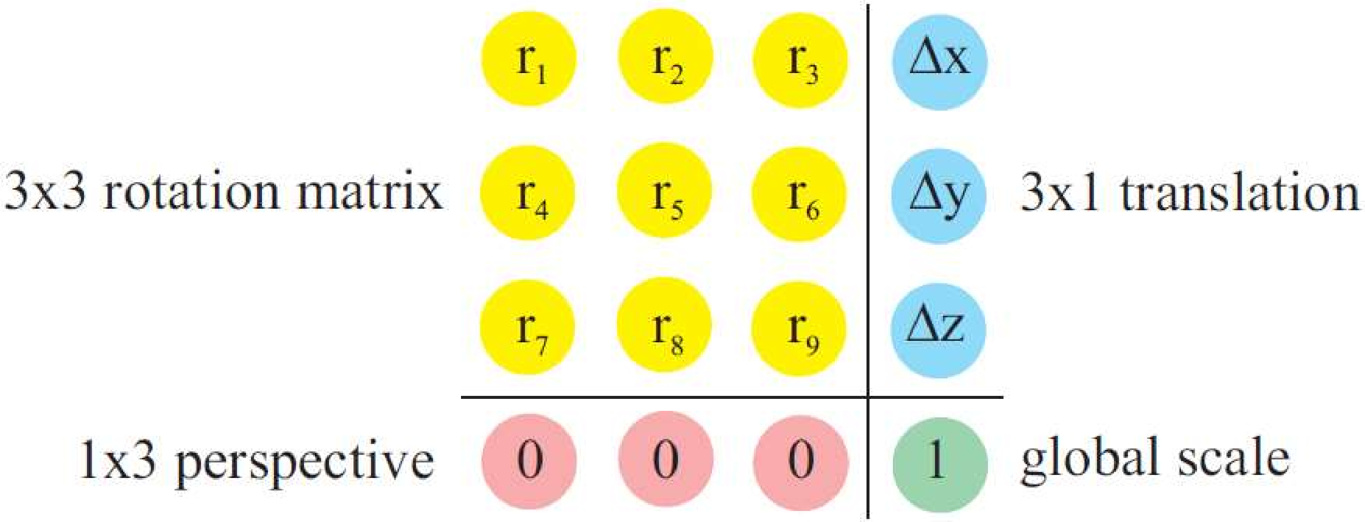
A more elegant form using the transformation matrix: ${}^{A}_{EE}T$

$$ {}^{A}_{EE}T = \begin{bmatrix} {}^{A}_{EE}R & {}^{A}P_{EE} \\ 000 & 1 \end{bmatrix} = \begin{bmatrix} 3\times 3 & 3\times 1 \\ 1\times 3 & 1\times 1 \end{bmatrix} = \begin{bmatrix} 4\times 4 \end{bmatrix} $$

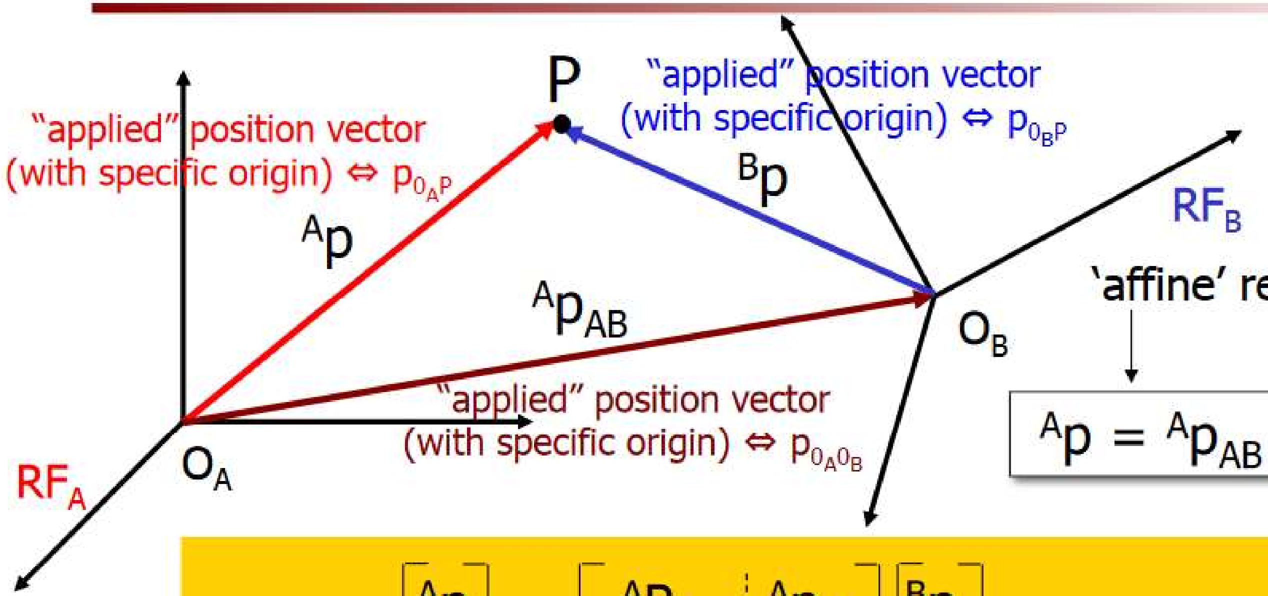
This is called ***<u>Homogeneous Transformation</u>*** and it will be useful when considering multiple frames of the robot and mapping the positions to arrive at the formulation of the kinematic problems

HOMOGENEUOS TRANSFORMATIONS

$$ {}_{EE}^{A}T = \begin{bmatrix} {}_{EE}^{A}R & {}^{A}P_{EE} \\ 000 & 1 \end{bmatrix} $$



Homogeneous transformations



'affine' relationship

$$^{A}p = {}^{A}p_{AB} + {}^{A}R_B \, {}^{B}p$$

$$^{A}p_{hom} = \begin{bmatrix} ^{A}p \\ \hline 1 \end{bmatrix} = \left[\begin{array}{c|c} ^{A}R_B & ^{A}p_{AB} \\ \hline 0\ 0\ 0 & 1 \end{array}\right] \begin{bmatrix} ^{B}p \\ \hline 1 \end{bmatrix} = {}^{A}T_B \, {}^{B}p_{hom}$$

← linear relationship

vector in homogeneous coordinates

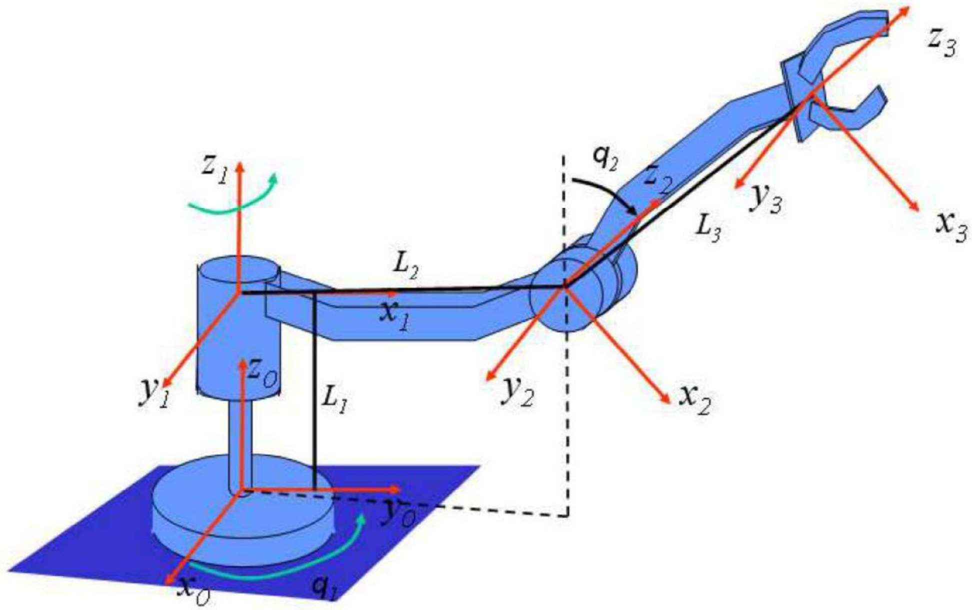
4x4 matrix of homogeneous transformation

Summary

Properties of T matrix

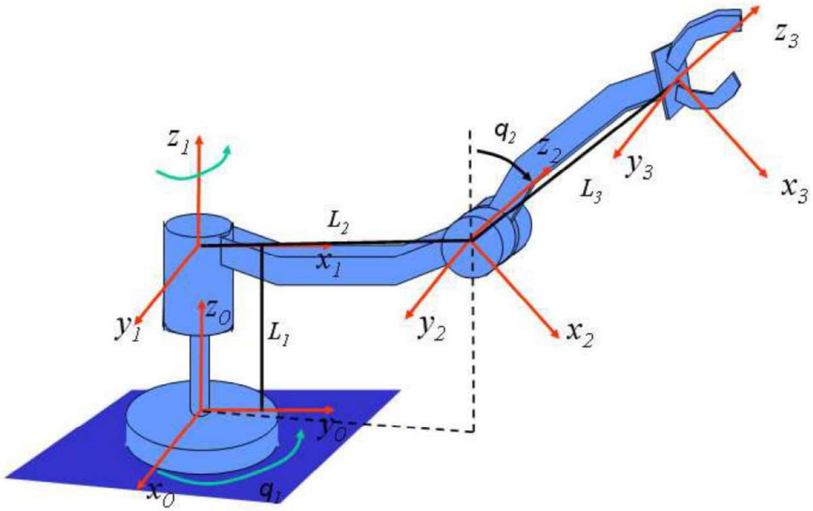
- describes the relation between reference frames (relative pose = position & orientation)
- transforms the representation of a position vector (applied vector starting from the origin of the frame) from a given frame to another frame
- it is a roto-translation operator on vectors in the three-dimensional space
- it is always invertible $({}^{A}T_{B})^{-1} = {}^{B}T_{A}$
- can be composed, i.e., ${}^{A}T_{C} = {}^{A}T_{B}\,{}^{B}T_{C}$ ← note: it does not commute!

Robot kinematics using homogeneous transformations



Knowing the *geometrical features* of a manipulator and associating to each link a *reference frame,* we are able to use *homogenous transformations* among the links and formulate the **<u>Kinematics</u>**

Robot kinematics using homogeneous transformations



Considering the four frames, arbitrary chosen on the four links, it is possible to write down the matrixes of the single homogeneous transformations.

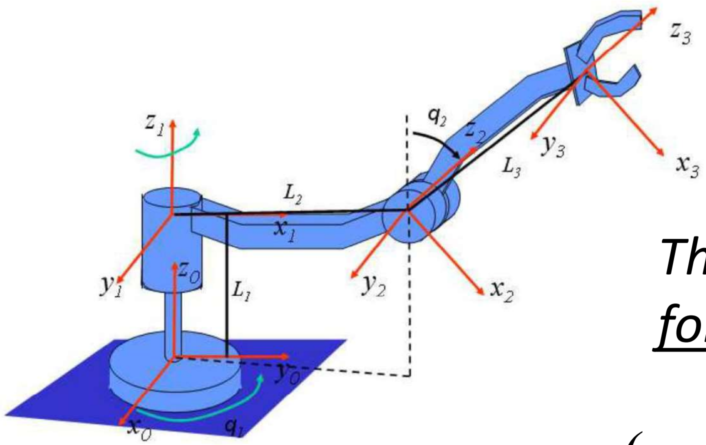
$$^{0}T_1 = Rot(z, q_1)Trasl(z, L_1);$$
$$^{1}T_2 = Trasl(x, L_2)Rot(y, q_2);$$
$$^{2}T_3 = Trasl(z, L_3);$$

Multiplication among the transformation leads to the transformation between the ***BASE {0}*** *and the* ***END-EFFECTOR {3}*** *expressed in function of the configuration space variables (**q_1** and **q_2**)*

$$^{0}T_3 = ^{0}T_1^{1}T_2^{2}T_3 = \left[\begin{array}{ccc|c} C_1C_2 & -S_1 & C_1S_2 & L_3S_2C_1 + L_2C_1 \\ S_1C_2 & C_1 & S_1S_2 & L_3S_2S_1 + L_2S_1 \\ -S_2 & 0 & C_2 & L_3C_2 + L_1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

Abbreviations: $C_i = \cos(q_i)$, $S_i = \sin(q_i)$

ROBOT KINEMATICS USING HOMOGENEOUS TRANSFORMATIONS

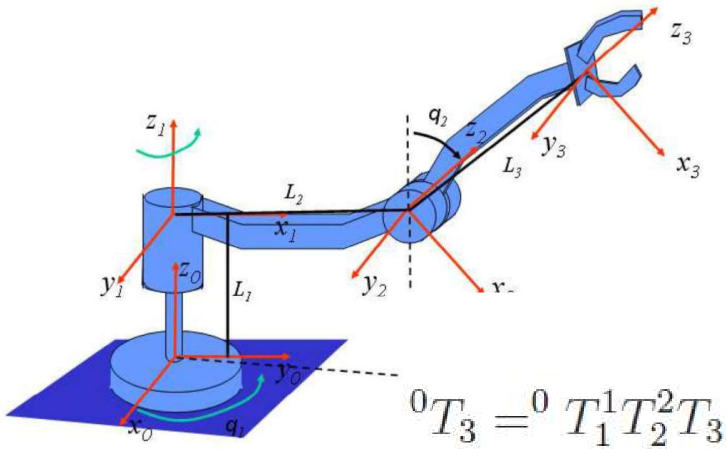


The obtained transformation is the *__forward kinematics__ of the manipulator.*

$$(x_1, x_2, x_3 \ldots . x_m) = F(q_1, q_2, q_3 \ldots .. q_n)$$

$$^0T_3 = {^0T_1}{^1T_2}{^2T_3} = \left[\begin{array}{ccc|c} C_1C_2 & -S_1 & C_1S_2 & L_3S_2C_1 + L_2C_1 \\ S_1C_2 & C_1 & S_1S_2 & L_3S_2S_1 + L_2S_1 \\ -S_2 & 0 & C_2 & L_3C_2 + L_1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

ROBOT KINEMATICS USING HOMOGENEOUS TRANSFORMATIONS



Rotation of the end-effector

$$^{0}T_3 = {}^{0}T_1^{1}T_2^{2}T_3 = \left[\begin{array}{ccc|c} C_1C_2 & -S_1 & C_1S_2 & L_3S_2C_1 + L_2C_1 \\ S_1C_2 & C_1 & S_1S_2 & L_3S_2S_1 + L_2S_1 \\ -S_2 & 0 & C_2 & L_3C_2 + L_1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

Coordinates of the end-effector respect to the base:

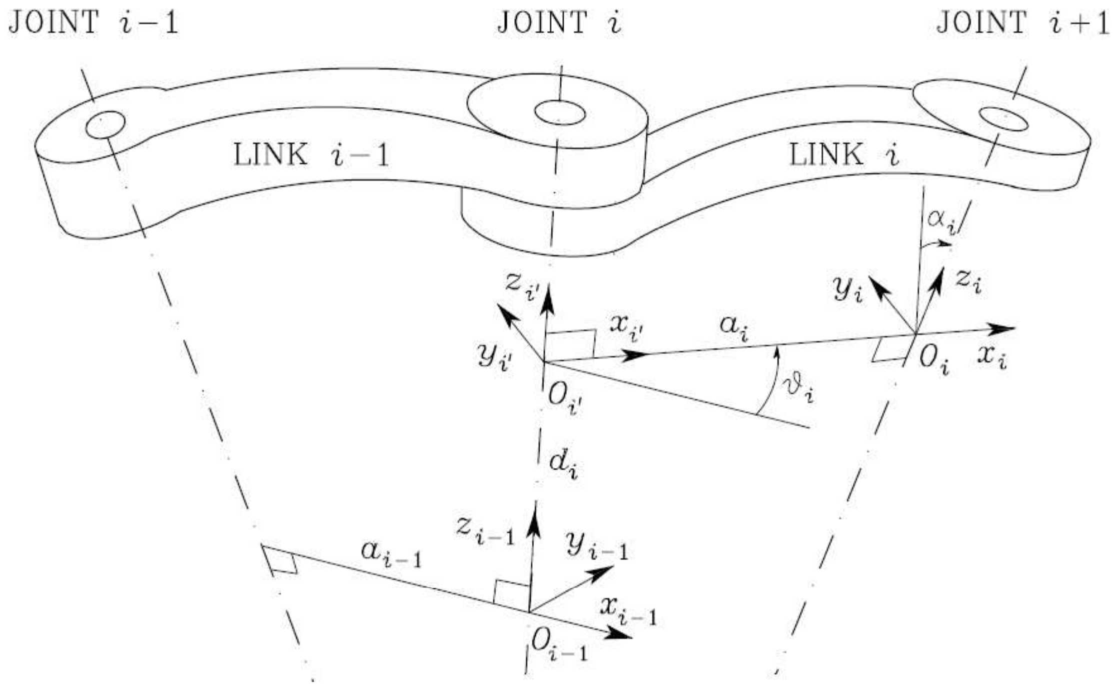
$$x = L_3 \sin \vartheta_2 \cos \vartheta_1 + L_2 \cos \vartheta_1$$

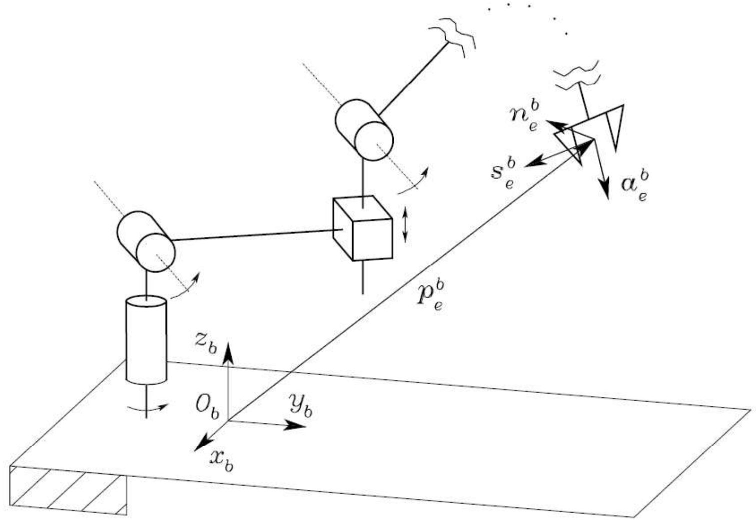
$$y = L_3 \sin \vartheta_2 \sin \vartheta_1 + L_2 \sin \vartheta_1$$

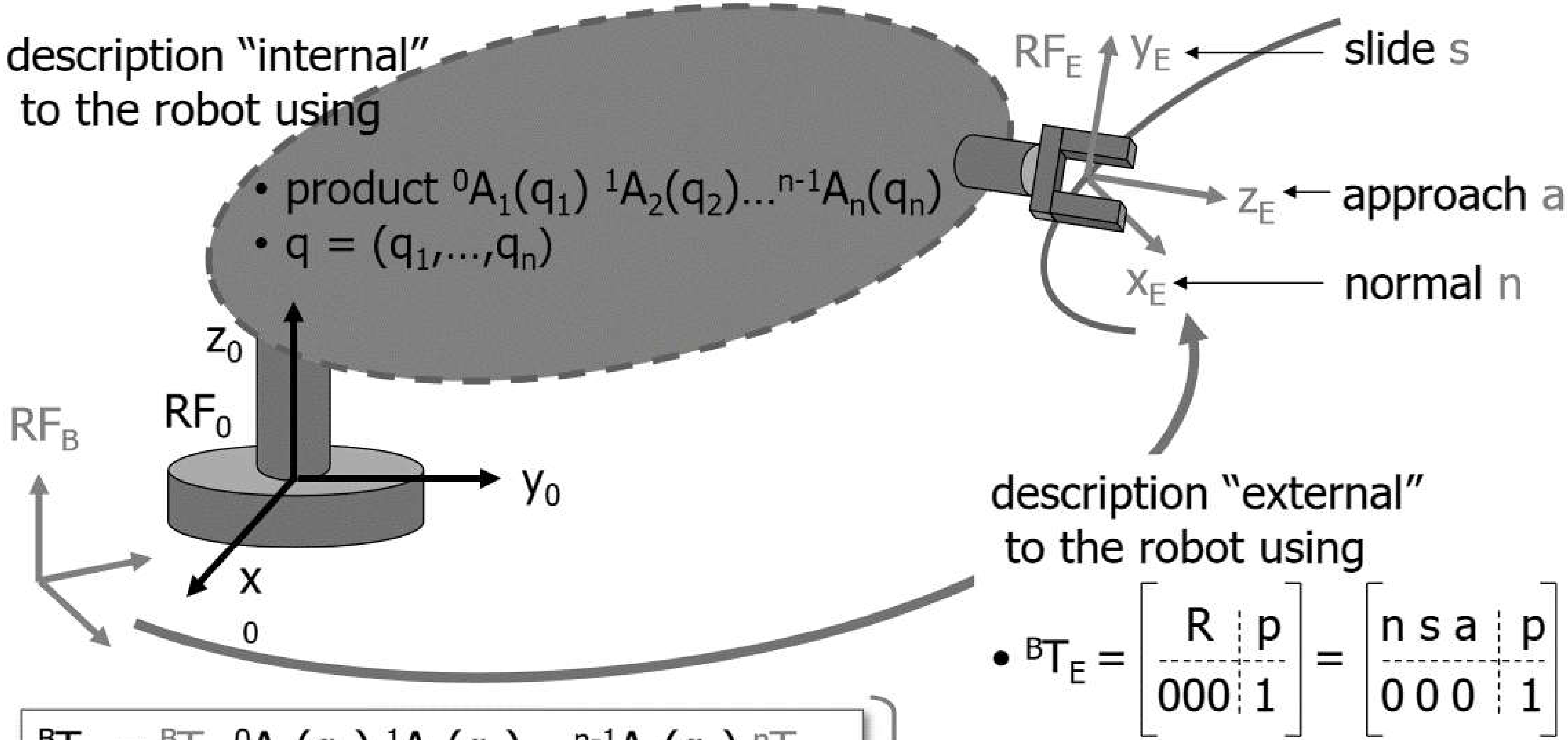
$$z = L_3 \cos \vartheta_2 + L_1$$

Denavit–Hartenberg Convention

A method is to be derived to define the relative position and orientation of two consecutive links;
the problem is that to determine two frames attached to the two links and compute the coordinate transformations between them.



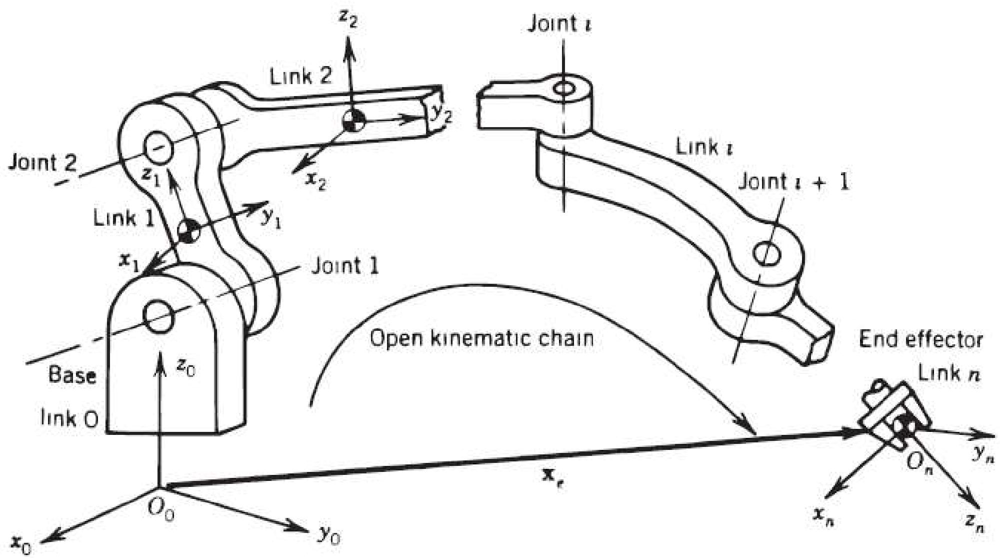




$$^{B}T_{E} = {}^{B}T_{0}\ {}^{0}A_{1}(q_{1})\ {}^{1}A_{2}(q_{2})\ \ldots {}^{n-1}A_{n}(q_{n})\ {}^{n}T_{E}$$

$$r = f_{r}(q)$$

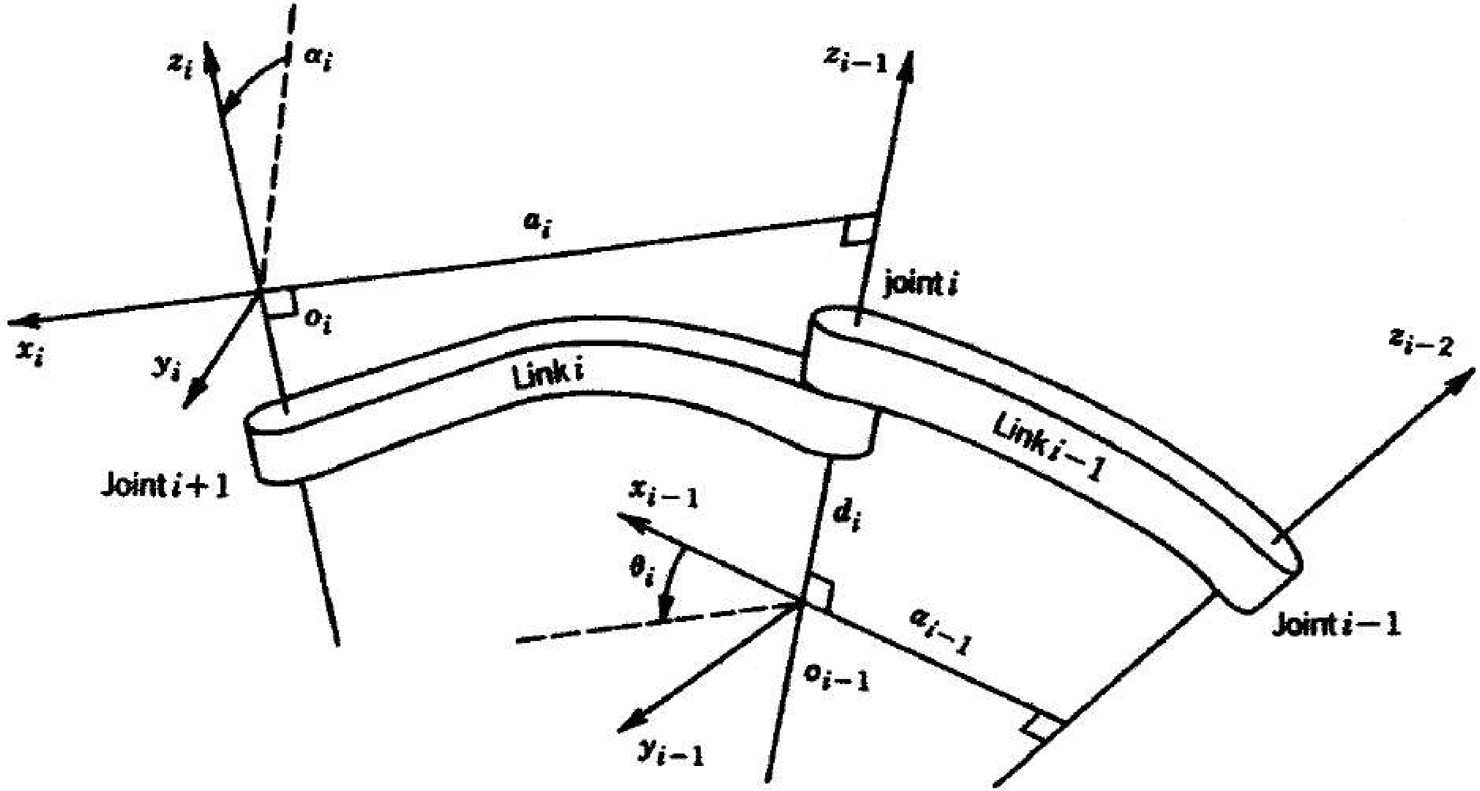
alternative descriptions of the **direct kinematics** of the robot the forward kinematics problem can be addressed using a more systematic method by the Denavit-Hartenberg convention.



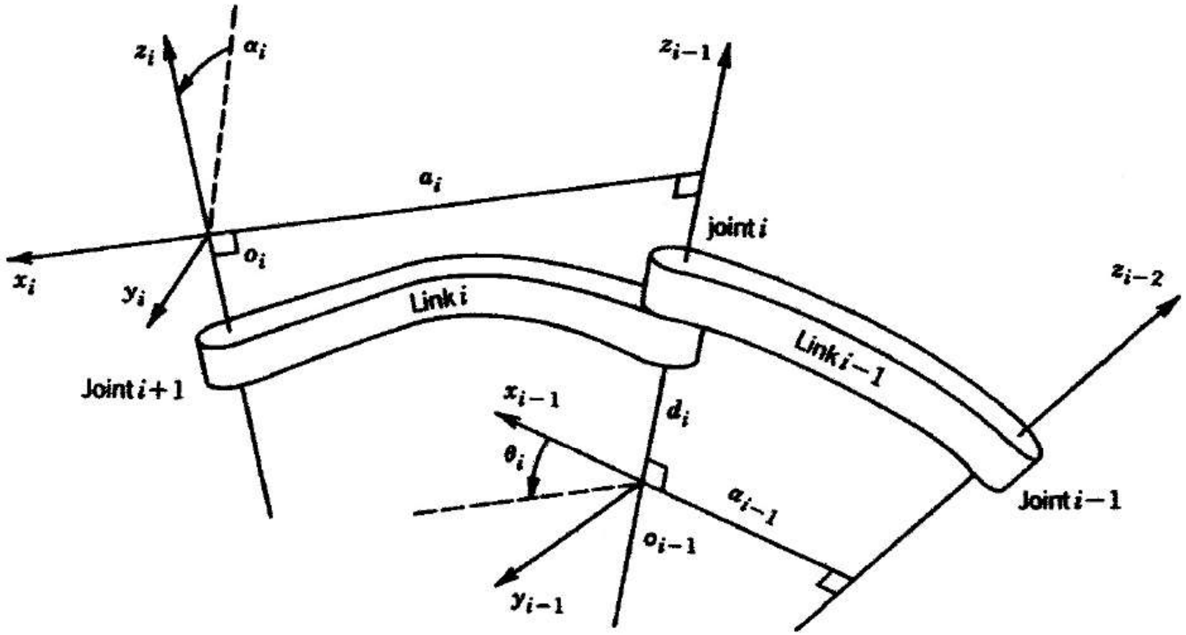
In *D-H convention* each link has a reference frame opportunely placed and with the orthonormal axes opportunely directed

THE DENAVIT-HARTENBERG CONVENTION

In *D-H convention* each homogeneous transformation between consecutive links is thought as **Four** consecutive transformations (link ***i*** in link ***i-1*** coordinates)



THE DENAVIT-HARTENBERG CONVENTION

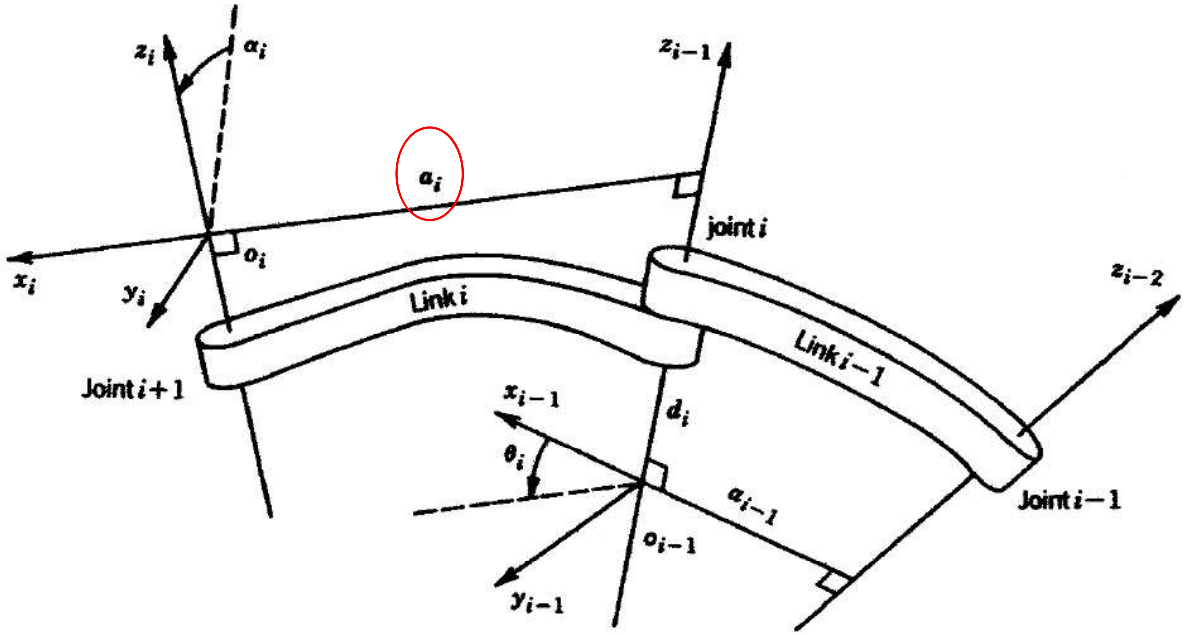


In **D-H convention** there are important 4 quantities:

1. Link length a_i
2. Link twist α_i
3. Link offset d_i
4. Joint angle θ_i

These quantities are always evaluated from the link ***i*** respect to the link ***i-1***

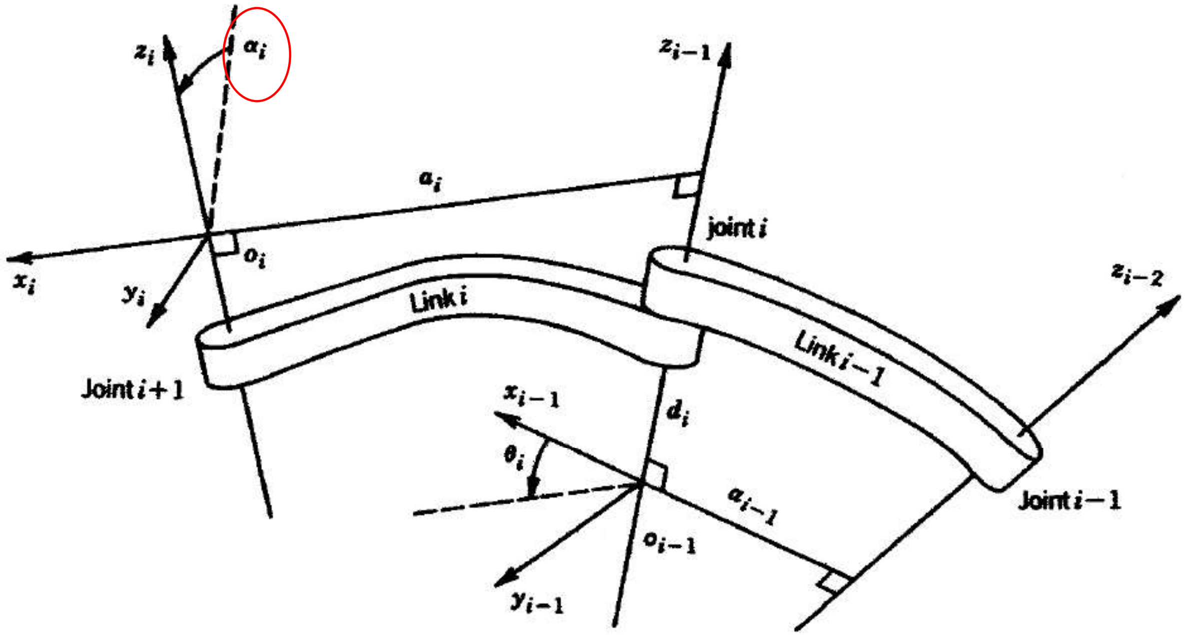
THE DENAVIT-HARTENBERG CONVENTION



<u>Link length</u> a_i

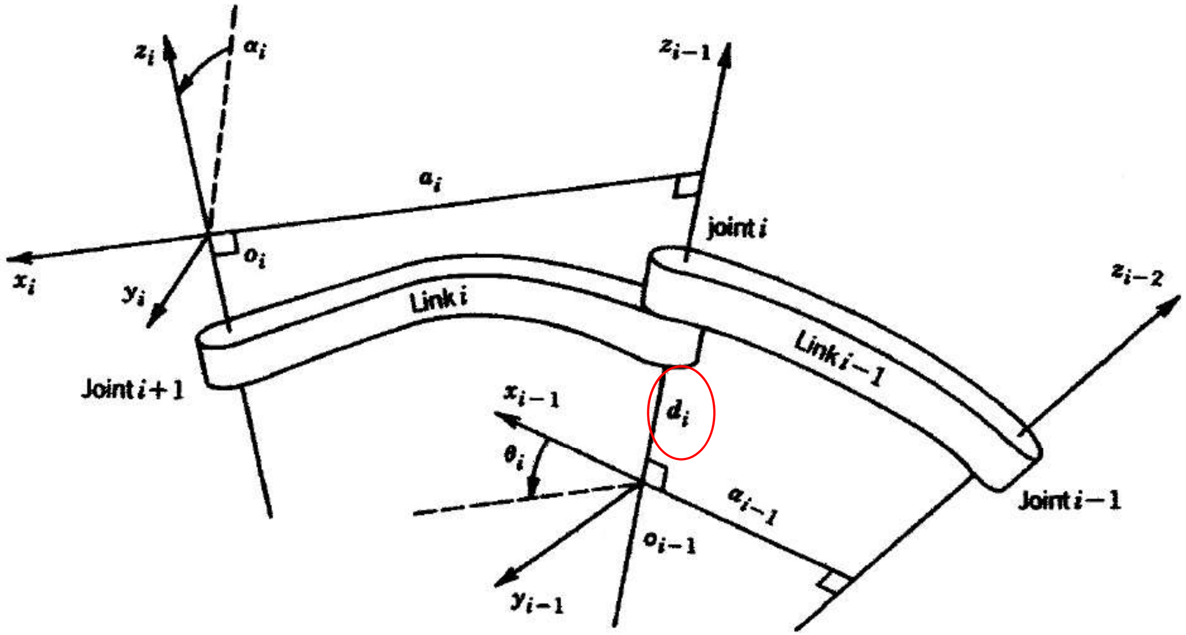
It is the distance between the axes Z_i and Z_{i-1} measured along x_i

THE DENAVIT-HARTENBERG CONVENTION



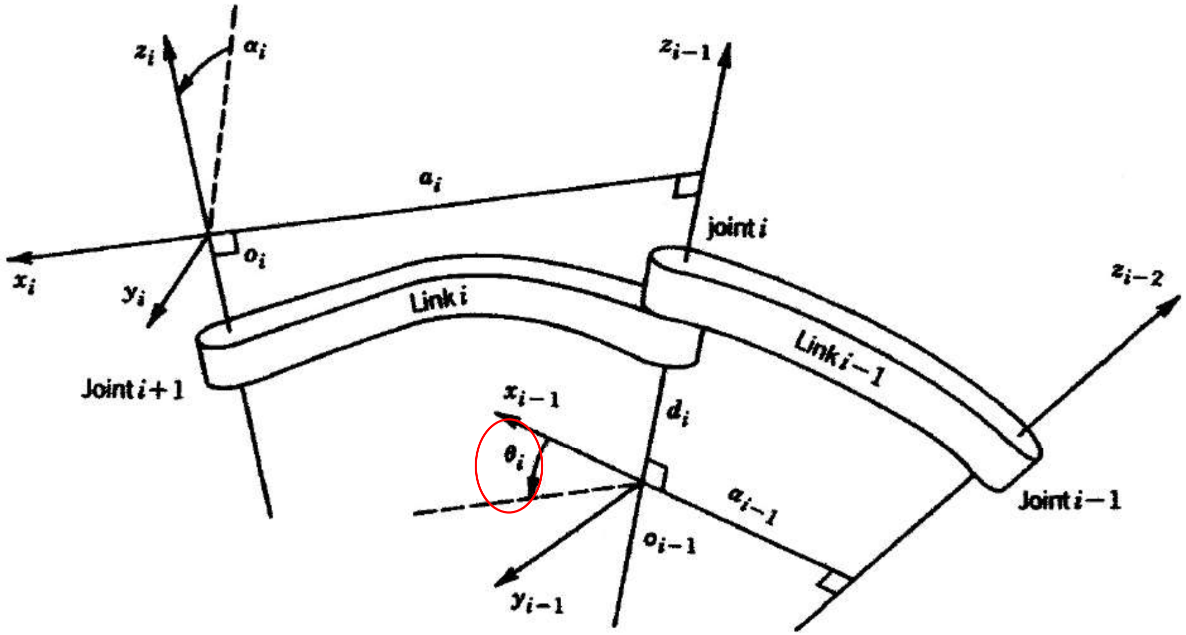
<u>Link twist</u> α_i

It is the angle between the axes ***Zi*** and ***Zi-1*** measured around ***xi***



<u>Link offset</u> d_i

It is the distance between the axes X_i and X_{i-1} measured along Z_{i-1}



Joint angle θ_i

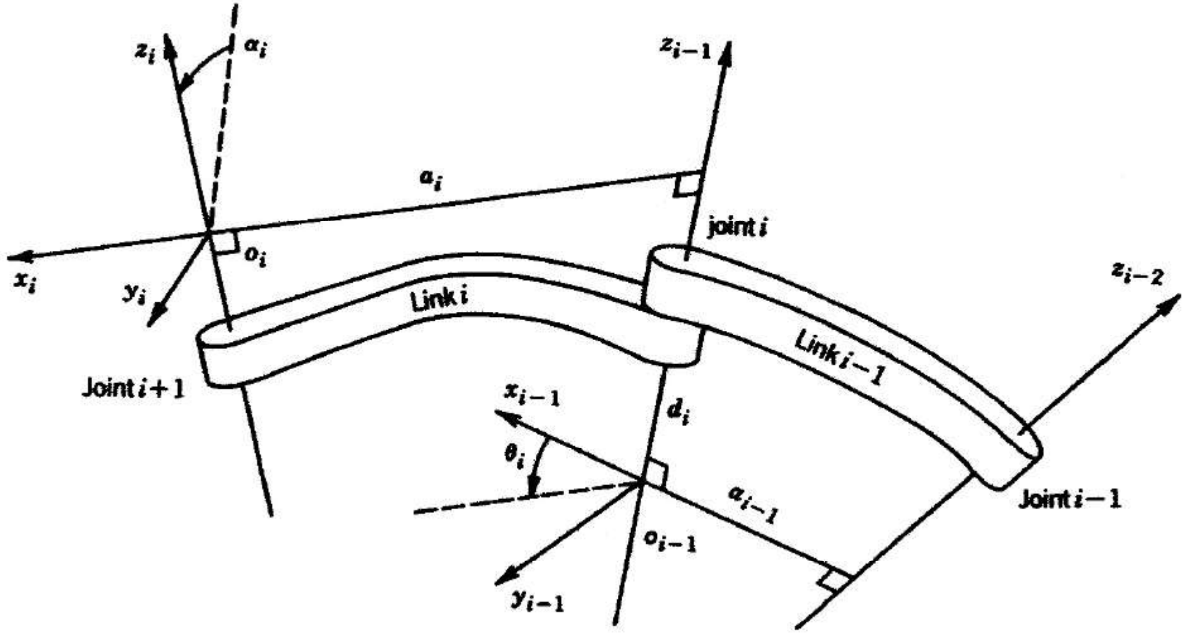
It is the angle between the axes X_i and X_{i-1} measured around Z_{i-1}

THE DENAVIT-HARTENBERG CONVENTION

The idea is composing a TABLE which will allow to find the homogeneous transformation among each pair of link

In **D-H convention** there are important 4 quantities:

1. Link length a_i
2. Link twist α_i
3. Link offset d_i
4. Joint angle θ_i



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D-H Table convention

Create a table of link parameters a_i, d_i, α_i, θ_i.

$a_i =$ distance along x_i from o_i to the intersection of the x_i and z_{i-1} axes.

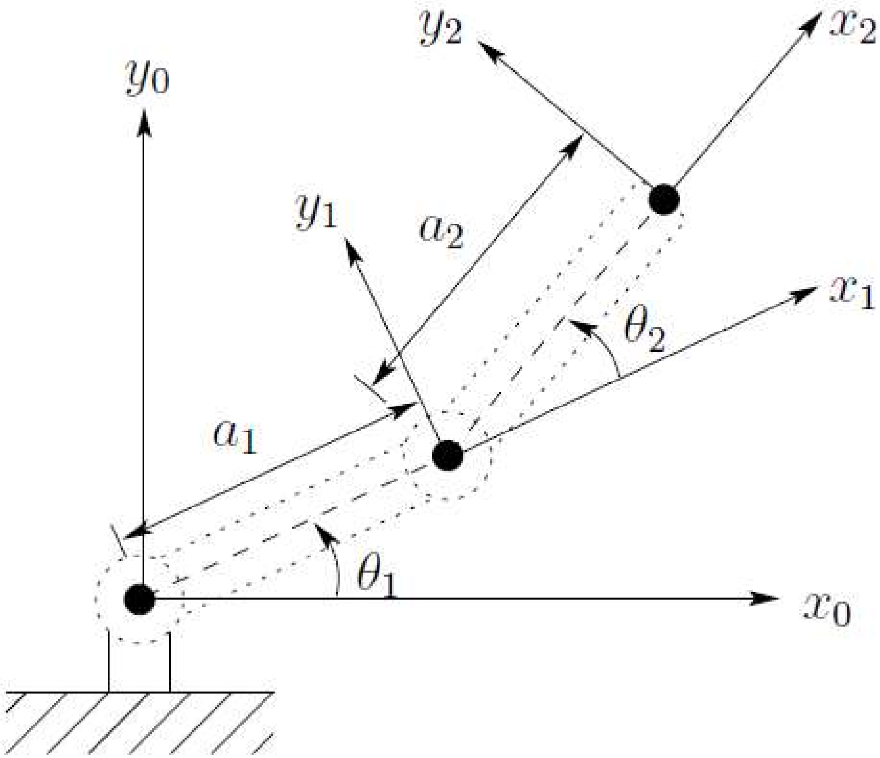
$d_i =$ distance along z_{i-1} from o_{i-1} to the intersection of the x_i and z_{i-1} axes. d_i is variable if joint i is prismatic.

$\alpha_i =$ the angle between z_{i-1} and z_i measured about x_i

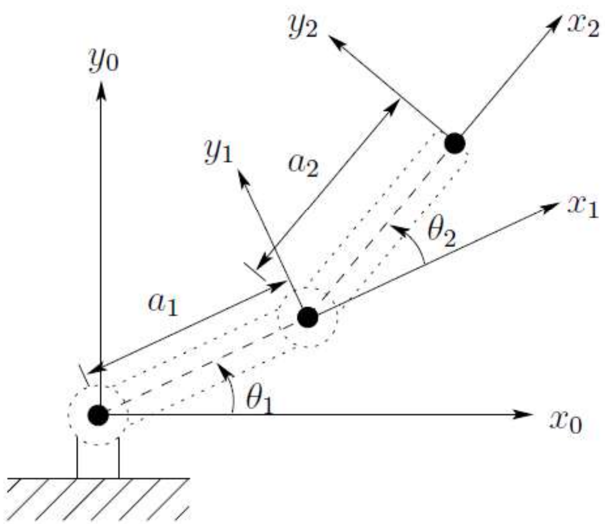
$\theta_i =$ the angle between x_{i-1} and x_i measured about z_{i-1} . θ_i is variable if joint i is revolute.

Link	a_i	α_i	d_i	θ_i
1				
2				
.				
n				

Example 1: Planar Elbow Manipulator



First row between link 0 and 1

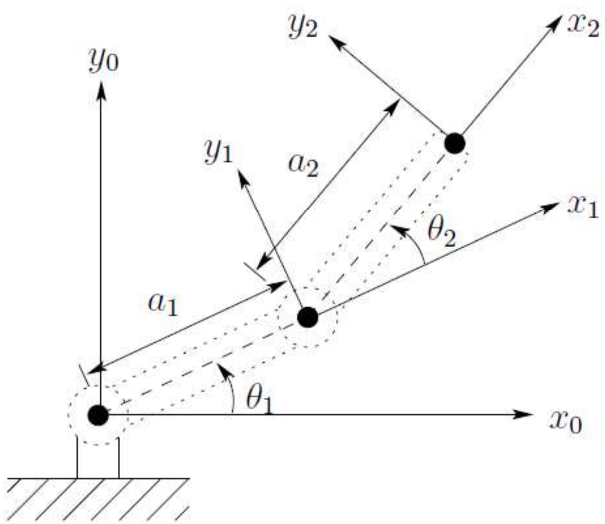


Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	θ_1^*
2	a_2	0	0	θ_2^*

<u>Link length</u> a_1

It the distance between the axes ***Z1*** and ***Z0*** measured along ***x1***

First row between link 0 and 1

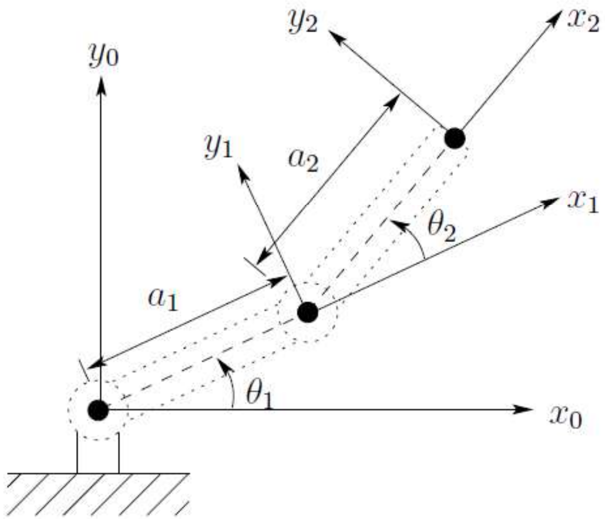


Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	θ_1^*
2	a_2	0	0	θ_2^*

<u>Link twist</u> α_1

It the angle between the axes Z_1 and Z_0 measured around X_1

First row between link 0 and 1

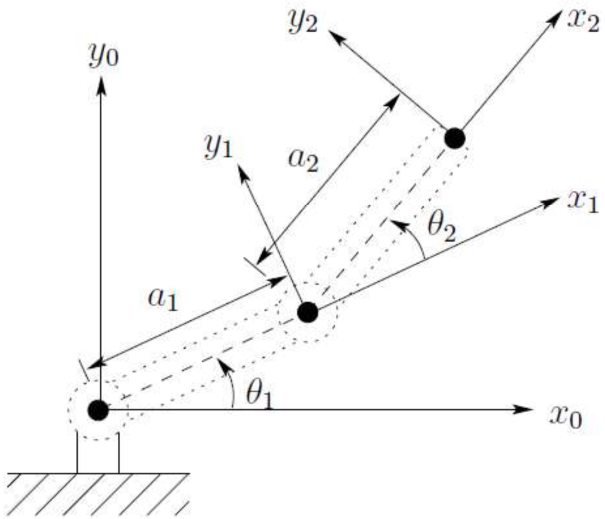


Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	θ_1^*
2	a_2	0	0	θ_2^*

Link offset d_i

It the distance between the axes X_1 and X_0 measured along Z_0

First row between link 0 and 1

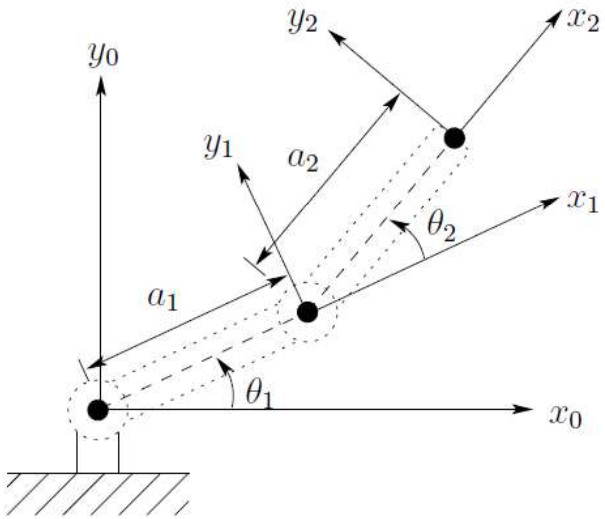


Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	θ_1^*
2	a_2	0	0	θ_2^*

Joint angle θ_i

It the angles between the axes ***X_1*** and ***X_0*** measured around ***Z_0***

second row between link 1 and 2

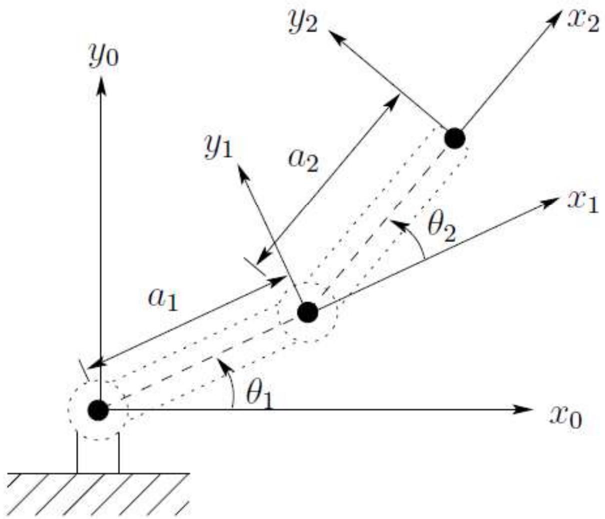


Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	θ_1^*
2	a_2	0	0	θ_2^*

<u>Link length</u> a_1

It the distance between the axes ***Z_2*** and ***Z_1*** measured along ***x_2***

second row between link 1 and 2

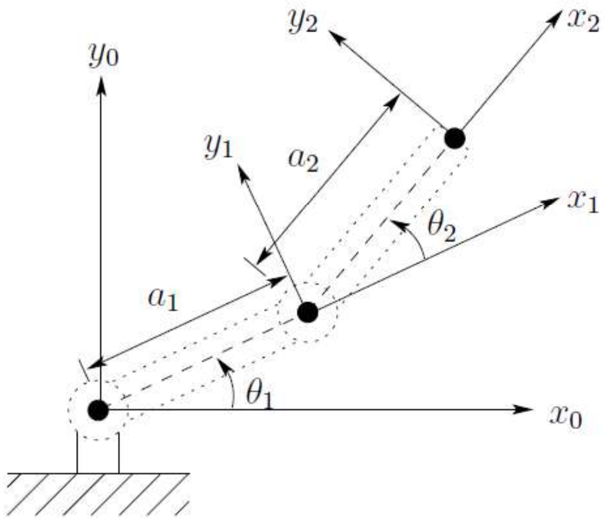


Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	θ_1^*
2	a_2	0	0	θ_2^*

<u>Link twist</u> α_1

It the angle between the axes Z_2 and Z_1 measured around X_2

second row between link 1 and 2

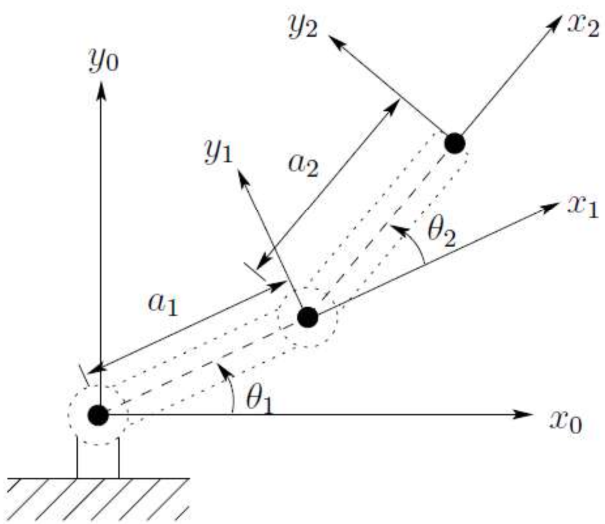


Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	θ_1^*
2	a_2	0	0	θ_2^*

Link offset d_i

It the distance between the axes ***X_2*** and ***X_1*** measured along ***Z_1***

second row between link 1 and 2

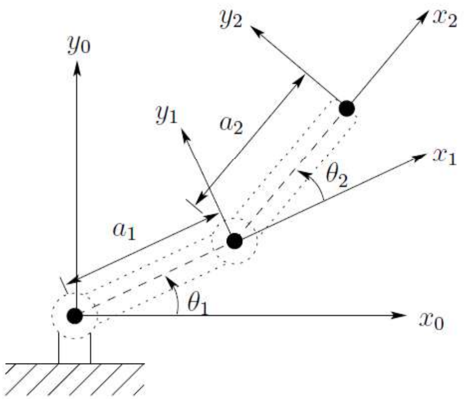


Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	θ_1^*
2	a_2	0	0	θ_2^*

Joint angle θ_i

It the angles between the axes ***X*2** and ***X*1** measured around ***Z*1**

We have now two homogeneous transformation



Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	θ_1^*
2	a_2	0	0	θ_2^*

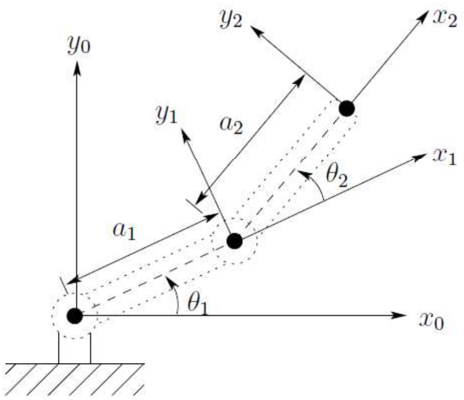
$$ {}^0T_1 = A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & a_1 c_1 \\ s_1 & c_1 & 0 & a_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Rotation around Z_0 and translation of a_1

$$ {}^1T_2 = A_2 = \begin{bmatrix} c_2 & -s_2 & 0 & a_2 c_2 \\ s_2 & c_2 & 0 & a_2 s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Rotation around Z_1 and translation of a_2

Finally the forward kinematics relating the base to the end effector



Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	θ_1^*
2	a_2	0	0	θ_2^*

Forward Kinematics is provided by the transformations:

$$^0T_2 = A_1 A_2 = \begin{bmatrix} c_{12} & -s_{12} & 0 & a_1 c_1 + a_2 c_{12} \\ s_{12} & c_{12} & 0 & a_1 s_1 + a_2 s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation of the end-effector around the axis z0

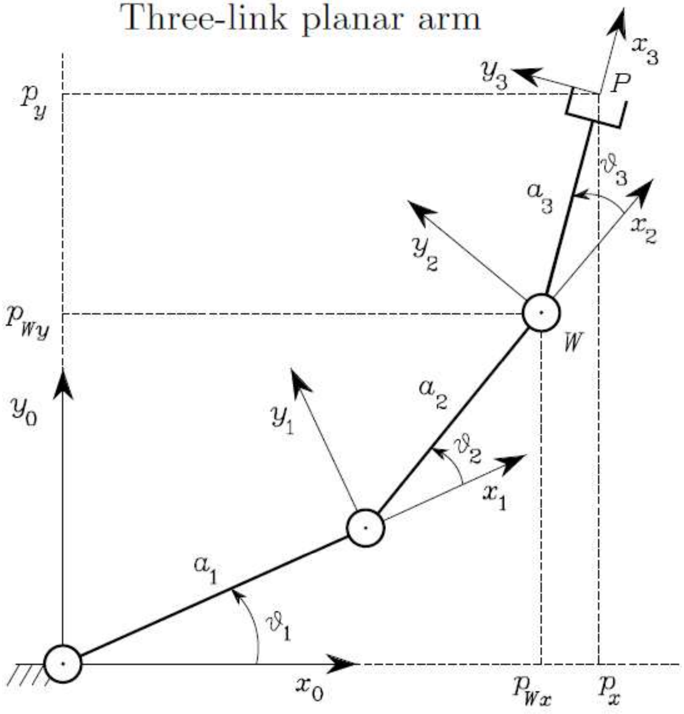
Coordinates of the end-effector respect to the base:

$$x = a_1 c_1 + a_2 c_{12}$$
$$y = a_1 s_1 + a_2 s_{12}$$

Kinematics of Typical Manipulator Structures (D-H)

Example 2: three link planar

Three-link planar arm



DH parameters for the three-link planar arm

Link	a_i	α_i	d_i	ϑ_i
1	a_1	0	0	ϑ_1
2	a_2	0	0	ϑ_2
3	a_3	0	0	ϑ_3

To REMEMBER

a_i --> distance Zi and Zi-1 along xi

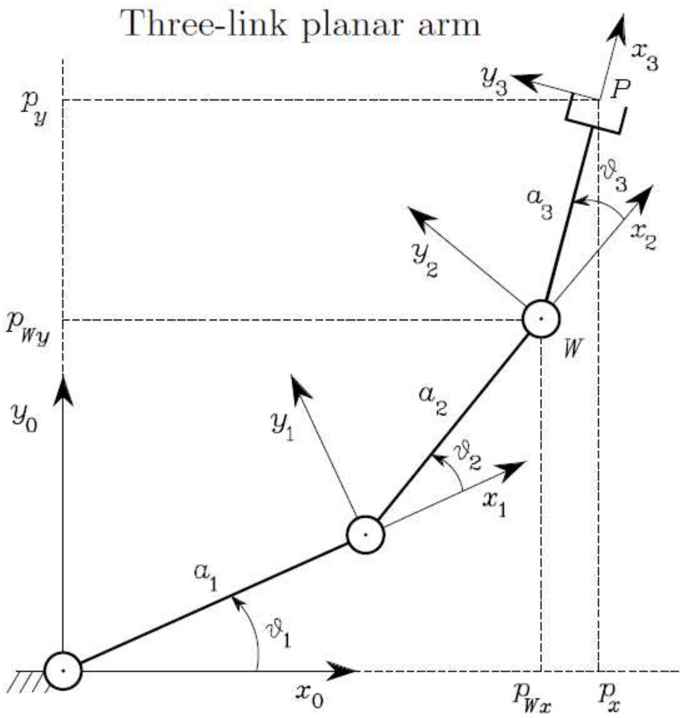
α_i -->angle Zi and Zi-1 around xi

d_i -->distance Xi and Xi-1 along Zi-1

θ_i --> angle Xi and Xi-1 around Zi-1

$$A_i^{i-1}(\vartheta_i) = \begin{bmatrix} c_i & -s_i & 0 & a_i c_i \\ s_i & c_i & 0 & a_i s_i \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Kinematics of Typical Manipulator Structures (D-H)



$$A_i^{i-1}(\vartheta_i) = \begin{bmatrix} c_i & -s_i & 0 & a_i c_i \\ s_i & c_i & 0 & a_i s_i \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad i = 1, 2, 3.$$

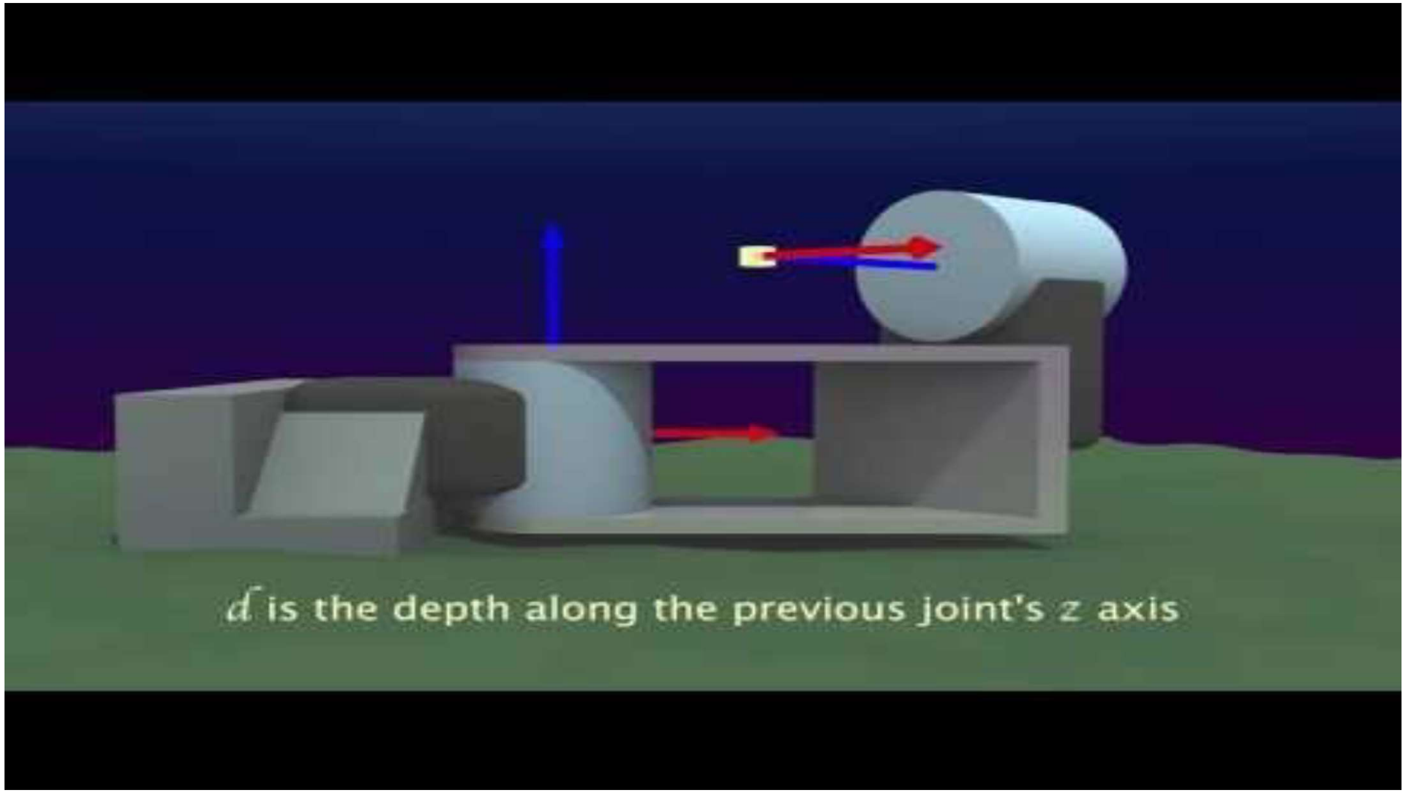
$$T_3^0(q) = A_1^0 A_2^1 A_3^2 = \begin{bmatrix} c_{123} & -s_{123} & 0 & a_1 c_1 + a_2 c_{12} + a_3 c_{123} \\ s_{123} & c_{123} & 0 & a_1 s_1 + a_2 s_{12} + a_3 s_{123} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$q = [\vartheta_1 \quad \vartheta_2 \quad \vartheta_3]^T$$

Rotation of the end-effector around the axis z0

Coordinates of the end-effector respect to the base

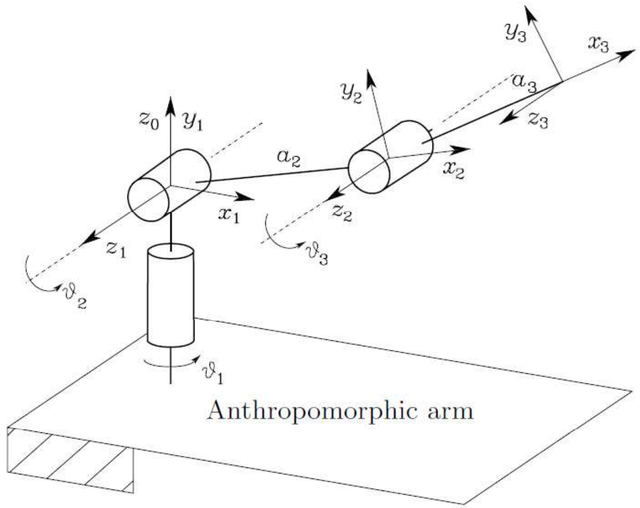
Video D-H: how to chose references

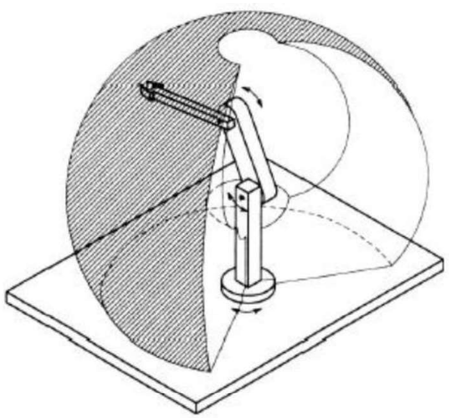


https://www.youtube.com/watch?v=rA9tm0gTIn8

Kinematics of Typical Manipulator Structures (D-H)

Example 3: antropomorphic Arm





DH parameters for the anthropomorphic arm

Link	a_i	α_i	d_i	ϑ_i
1	0	$\pi/2$	0	ϑ_1
2	a_2	0	0	ϑ_2
3	a_3	0	0	ϑ_3

To REMEMBER

a_i --> distance Zi and Zi-1 along xi

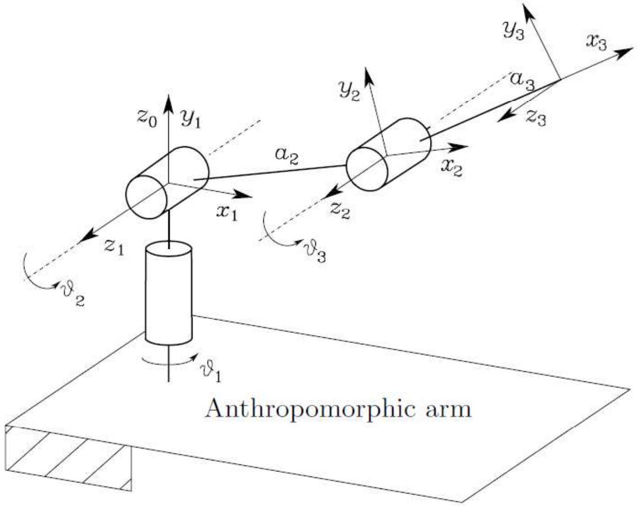
α_i -->angle Zi and Zi-1 around xi

d_i -->distance Xi and Xi-1 along Zi-1

θ_i --> angle Xi and Xi-1 around Zi-1

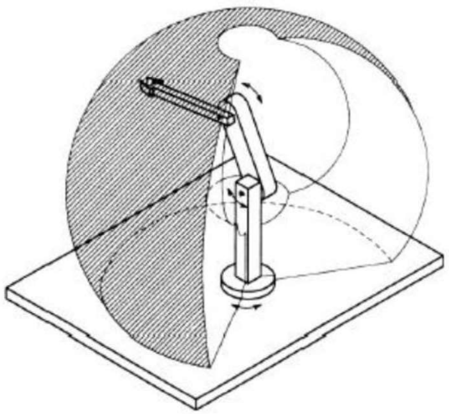
Kinematics of Typical Manipulator Structures (D-H)

Example 3: antropomorphic Arm



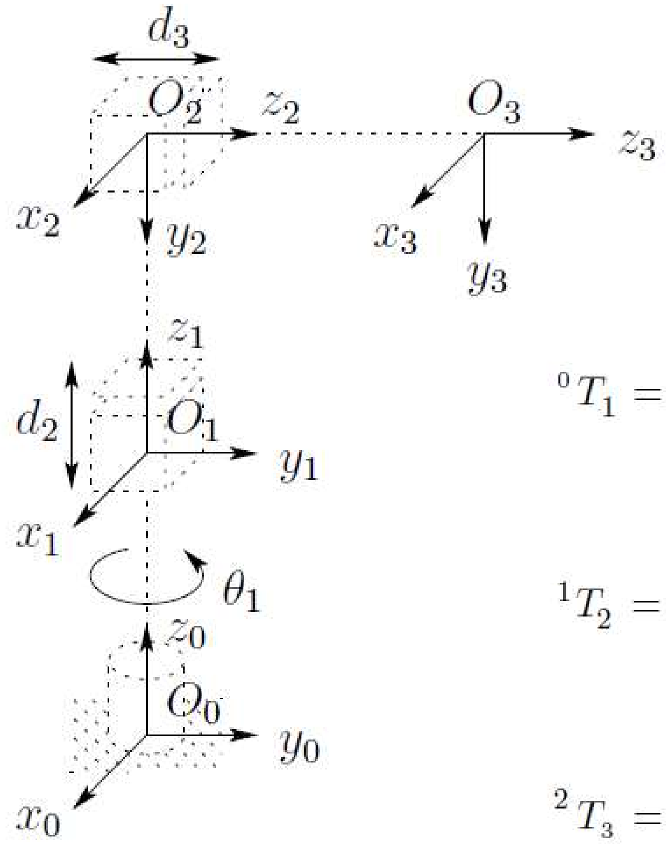
$$\boldsymbol{A}_1^0(\vartheta_1) = \begin{bmatrix} c_1 & 0 & s_1 & 0 \\ s_1 & 0 & -c_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{A}_i^{i-1}(\vartheta_i) = \begin{bmatrix} c_i & -s_i & 0 & a_i c_i \\ s_i & c_i & 0 & a_i s_i \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad i = 2, 3.$$



$$\boldsymbol{T}_3^0(\boldsymbol{q}) = \boldsymbol{A}_1^0 \boldsymbol{A}_2^1 \boldsymbol{A}_3^2 = \begin{bmatrix} c_1 c_{23} & -c_1 s_{23} & s_1 & c_1(a_2 c_2 + a_3 c_{23}) \\ s_1 c_{23} & -s_1 s_{23} & -c_1 & s_1(a_2 c_2 + a_3 c_{23}) \\ s_{23} & c_{23} & 0 & a_2 s_2 + a_3 s_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 4: Three-Link Cylindrical Robot



Link	a_i	α_i	d_i	θ_i
1	0	0	d_1	θ_1^*
2	0	−90	d_2^*	0
3	0	0	d_3^*	0

* variable

$$^0T_1 = A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^1T_2 = A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^2T_3 = A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

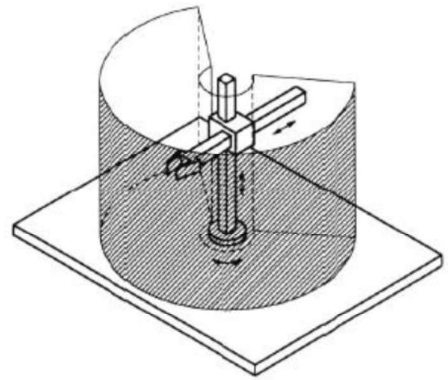
To REMEMBER

$\boldsymbol{a_i}$ --> distance Zi and Zi-1 along xi

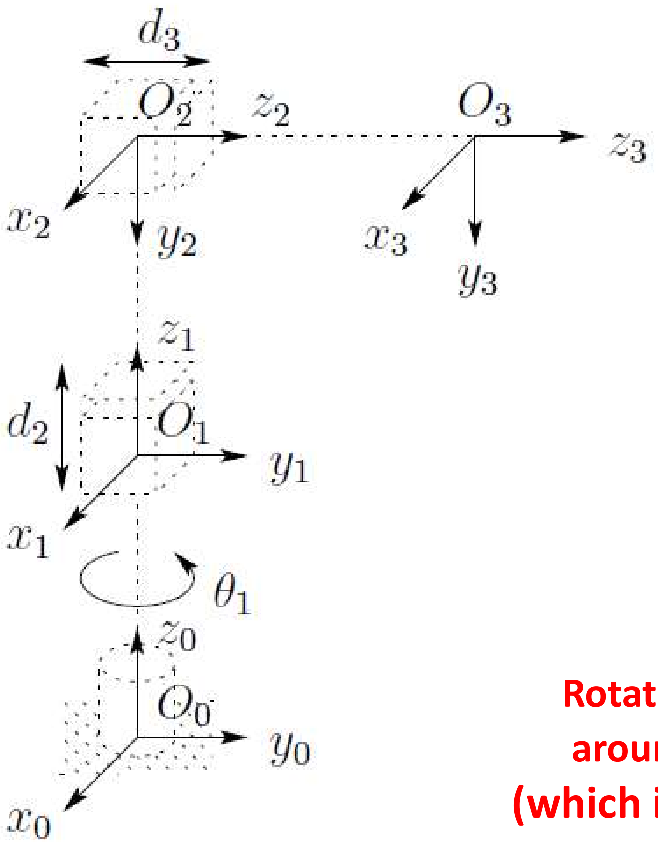
$\boldsymbol{\alpha_i}$ -->angle Zi and Zi-1 around xi

$\boldsymbol{d_i}$ -->distance Xi and Xi-1 along Zi-1

$\boldsymbol{\theta_i}$ --> angle Xi and Xi-1 around Zi-1



Example 4: Three-Link Cylindrical Robot



Link	a_i	α_i	d_i	θ_i
1	0	0	d_1	θ_1^*
2	0	-90	d_2^*	0
3	0	0	d_3^*	0

* variable

Forward Kinematics is provided by the transformations:

$$T_3^0 = A_1 A_2 A_3 = \begin{bmatrix} c_1 & 0 & -s_1 & -s_1 d_3 \\ s_1 & 0 & c_1 & c_1 d_3 \\ 0 & -1 & 0 & d_1 + d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation of the end-effector around the frame x0 y0 z0 (which is a rotation around z0)

Coordinates of the end-effector respect to the base

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Recalling Rotation matrix

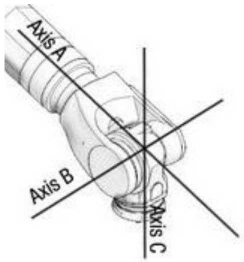
$$\boldsymbol{R}_z(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

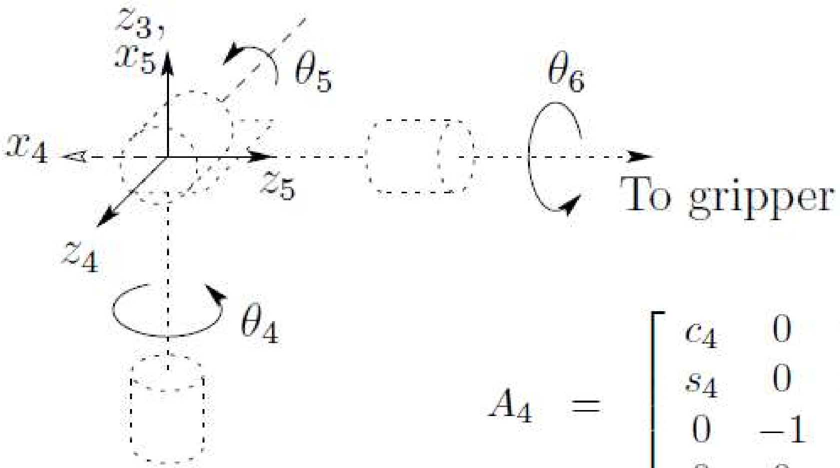
$$\boldsymbol{R}_y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$

$$\boldsymbol{R}_x(\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix}$$

Example 5: Spherical Wrist

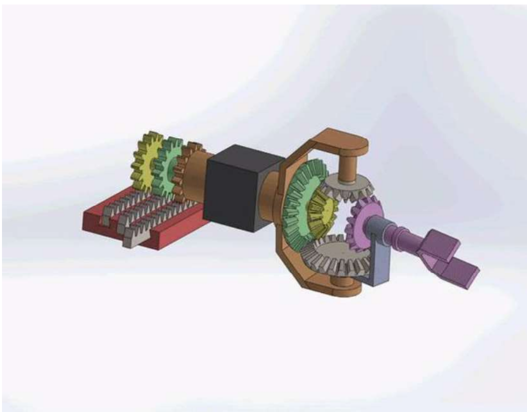
Assume in this case the base as the link 3 is not visible, and compute normally D-H.





Link	a_i	α_i	d_i	θ_i
4	0	−90	0	θ_4^*
5	0	90	0	θ_5^*
6	0	0	d_6	θ_6^*

* variable



$$A_4 = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_5 = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_6 = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

To REMEMBER

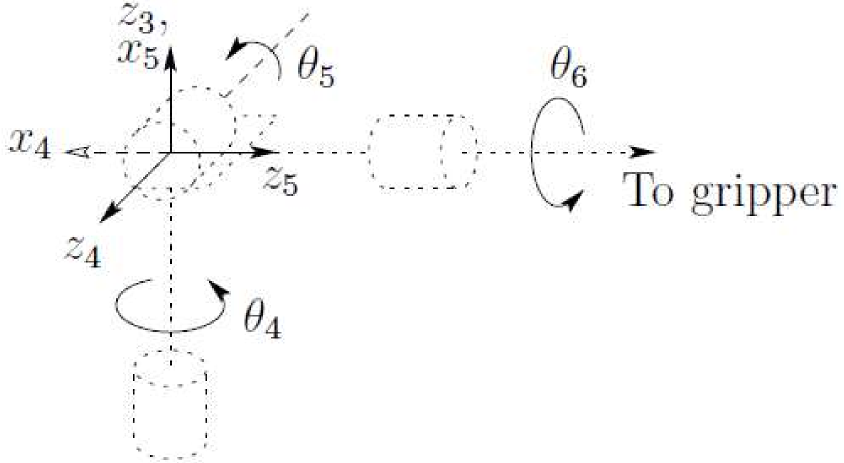
a_i --> distance Zi and Zi-1 along xi

α_i -->angle Zi and Zi-1 around xi

d_i -->distance Xi and Xi-1 along Zi-1

θ_i --> angle Xi and Xi-1 around Zi-1

Example 5: Spherical Wrist



Link	a_i	α_i	d_i	θ_i
4	0	−90	0	θ_4^*
5	0	90	0	θ_5^*
6	0	0	d_6	θ_6^*

* variable

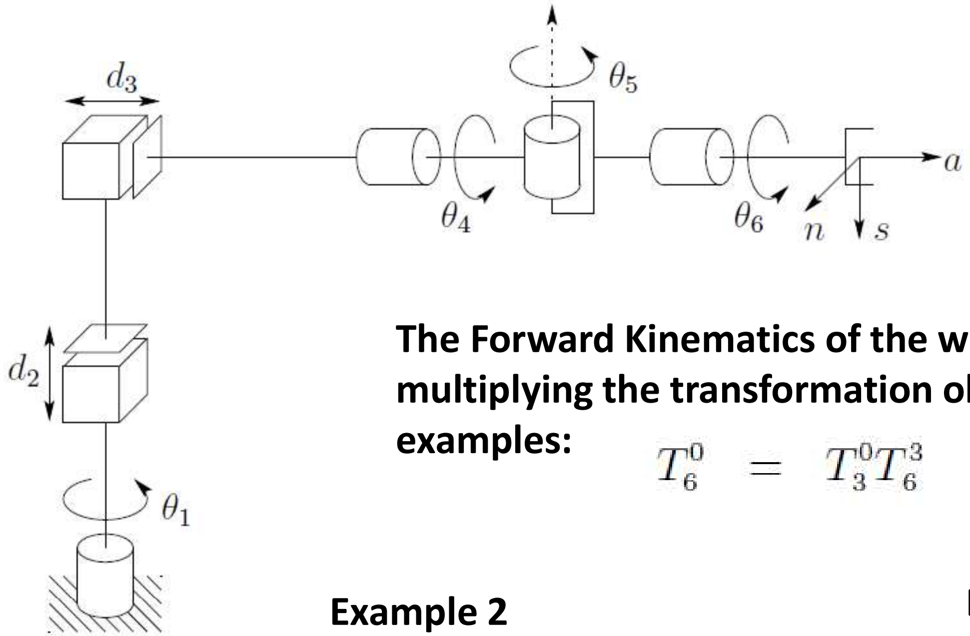
Coordinates of the end-effector respect to the base (in this case is link 3 the base which is not visible)

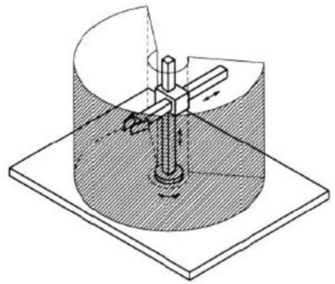
$$T_6^3 = A_4 A_5 A_6 = \begin{bmatrix} R_6^3 & o_6^3 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 & c_4s_5d_6 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 & s_4s_5d_6 \\ -s_5c_6 & s_5s_6 & c_5 & c_5d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

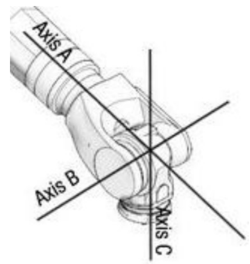
Rotation of the end-effector around the frame x4 y4 z4

Example 6: Cylindrical Manipulator + Spherical Wrist (Example 4-5 assembled)





+



The Forward Kinematics of the whole device is obtained by multiplying the transformation obtained in the previous examples:

$$T_6^0 = T_3^0 T_6^3$$

Example 2 **Example 3**

$$T_6^0 = \begin{bmatrix} c_1 & 0 & -s_1 & -s_1 d_3 \\ s_1 & 0 & c_1 & c_1 d_3 \\ 0 & -1 & 0 & d_1 + d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 & c_4s_5d_6 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 & s_4s_5d_6 \\ -s_5c_6 & s_5c_6 & c_5 & c_5d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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Example 6: Cylindrical Manipulator with Spherical Wrist (Example 2-3 assembled)

$$T_6^0 = \begin{bmatrix} c_1 & 0 & -s_1 & -s_1 d_3 \\ s_1 & 0 & c_1 & c_1 d_3 \\ 0 & -1 & 0 & d_1 + d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_4 c_5 c_6 - s_4 s_6 & -c_4 c_5 s_6 - s_4 c_6 & c_4 s_5 & c_4 s_5 d_6 \\ s_4 c_5 c_6 + c_4 s_6 & -s_4 c_5 s_6 + c_4 c_6 & s_4 s_5 & s_4 s_5 d_6 \\ -s_5 c_6 & s_5 c_6 & c_5 & c_5 d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation of the end-effector respect the frame x0 y0 z0

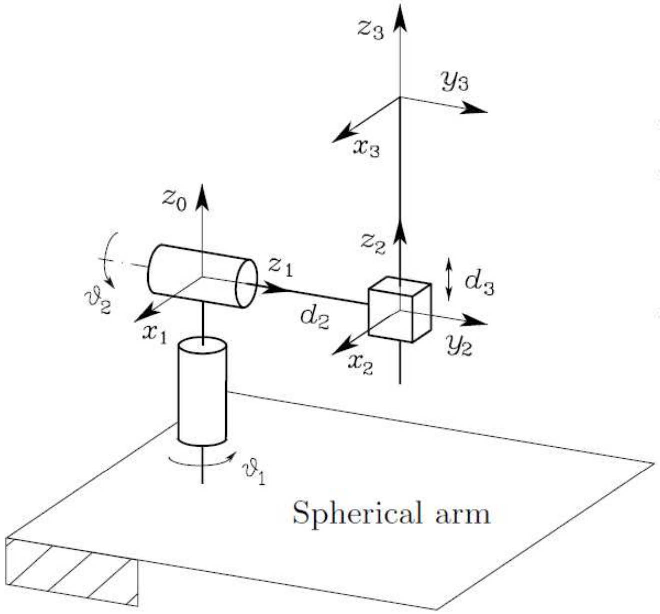
$$\begin{aligned}
r_{11} &= c_1 c_4 c_5 c_6 - c_1 s_4 s_6 + s_1 s_5 c_6 \\
r_{21} &= s_1 c_4 c_5 c_6 - s_1 s_4 s_6 - c_1 s_5 c_6 \\
r_{31} &= -s_4 c_5 c_6 - c_4 s_6 \\
r_{12} &= -c_1 c_4 c_5 s_6 - c_1 s_4 c_6 - s_1 s_5 c_6 \\
r_{22} &= -s_1 c_4 c_5 s_6 - s_1 s_4 s_6 + c_1 s_5 c_6 \\
r_{32} &= s_4 c_5 c_6 - c_4 c_6 \\
r_{13} &= c_1 c_4 s_5 - s_1 c_5 \\
r_{23} &= s_1 c_4 s_5 + c_1 c_5 \\
r_{33} &= -s_4 s_5
\end{aligned}$$

Coordinates of the end-effector respect to the base

$$\begin{aligned}
d_x &= c_1 c_4 s_5 d_6 - s_1 c_5 d_6 - s_1 d_3 \\
d_y &= s_1 c_4 s_5 d_6 + c_1 c_5 d_6 + c_1 d_3 \\
d_z &= -s_4 s_5 d_6 + d_1 + d_2.
\end{aligned}$$

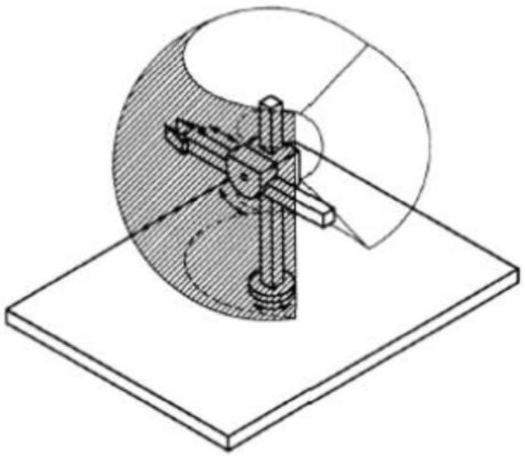
Kinematics of Typical Manipulator Structures (D-H)

Example 7: spherical Arm



DH parameters for the spherical arm

Link	a_i	α_i	d_i	ϑ_i
1	0	$-\pi/2$	0	ϑ_1
2	0	$\pi/2$	d_2	ϑ_2
3	0	0	d_3	0



To REMEMBER

a_i --> distance Zi and Zi-1 along xi

α_i -->angle Zi and Zi-1 around xi

d_i -->distance Xi and Xi-1 along Zi-1

θ_i --> angle Xi and Xi-1 around Zi-1

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Recalling Rotation matrix

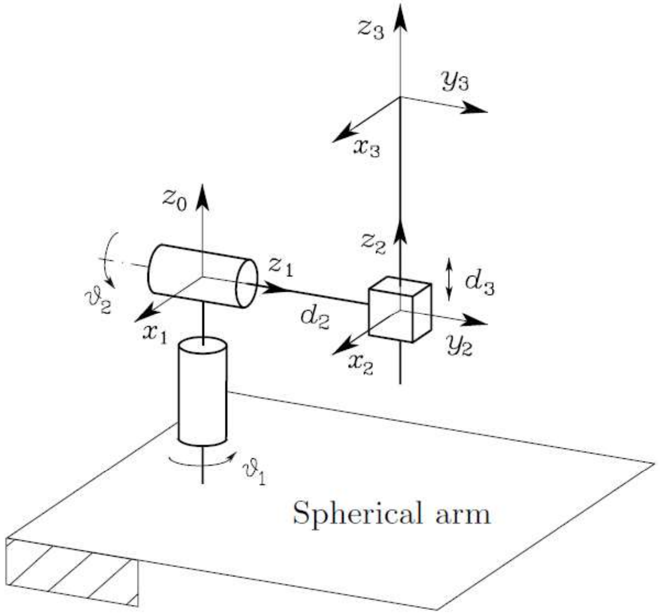
$$\boldsymbol{R}_z(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{R}_y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$

$$\boldsymbol{R}_x(\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix}$$

Kinematics of Typical Manipulator Structures (D-H)

Example 7: spherical Arm



$$\boldsymbol{A}_1^0(\vartheta_1) = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \boldsymbol{A}_2^1(\vartheta_2) = \begin{bmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{A}_3^2(d_3) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$\boldsymbol{T}_3^0(\boldsymbol{q}) = \boldsymbol{A}_1^0 \boldsymbol{A}_2^1 \boldsymbol{A}_3^2 = \begin{bmatrix} c_1 c_2 & -s_1 & c_1 s_2 & c_1 s_2 d_3 - s_1 d_2 \\ s_1 c_2 & c_1 & s_1 s_2 & s_1 s_2 d_3 + c_1 d_2 \\ -s_2 & 0 & c_2 & c_2 d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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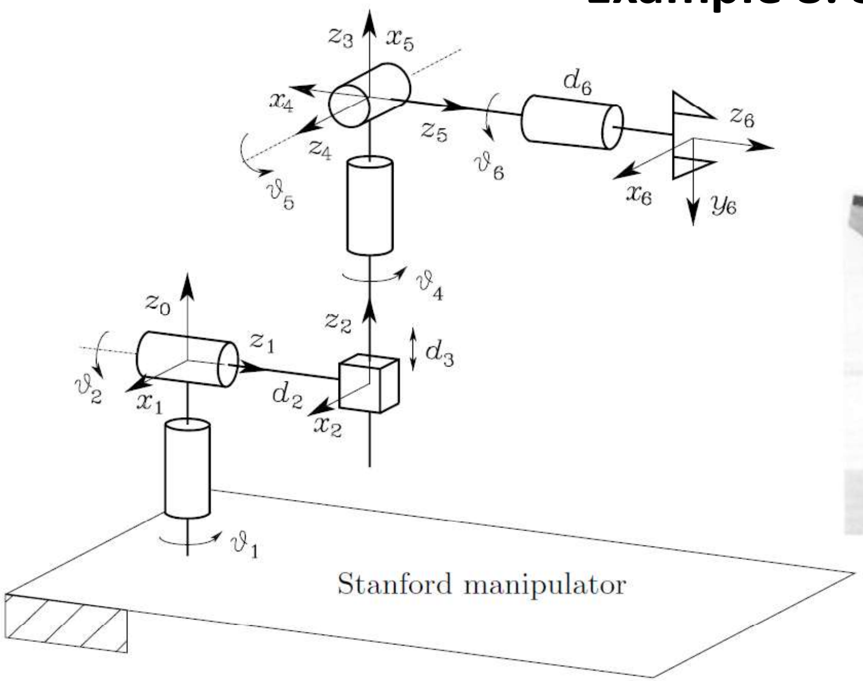
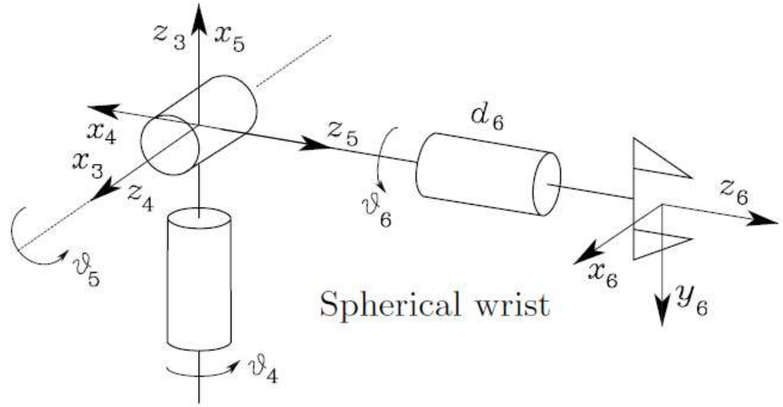
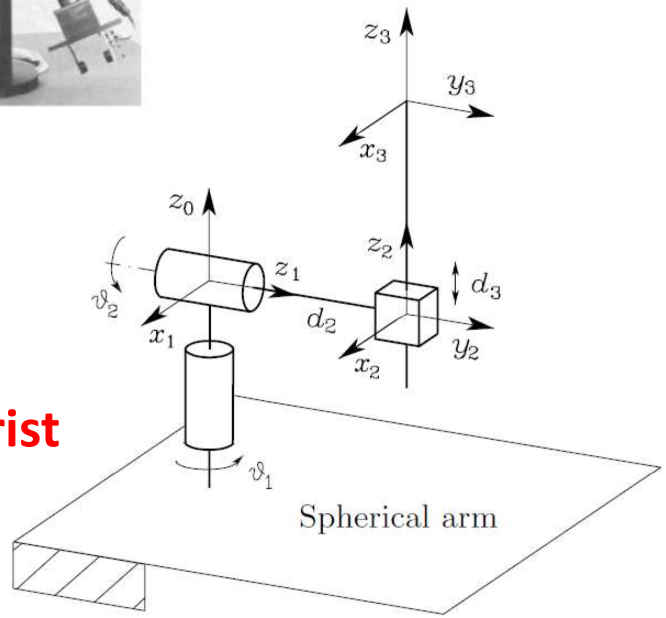
Kinematics of Typical Manipulator Structures (D-H)

Example 8: Stanford Manipulator







Stanford Manipulator:

it consists in a Spherical Arm + Spherical Wrist

Kinematics of Typical Manipulator Structures (D-H)

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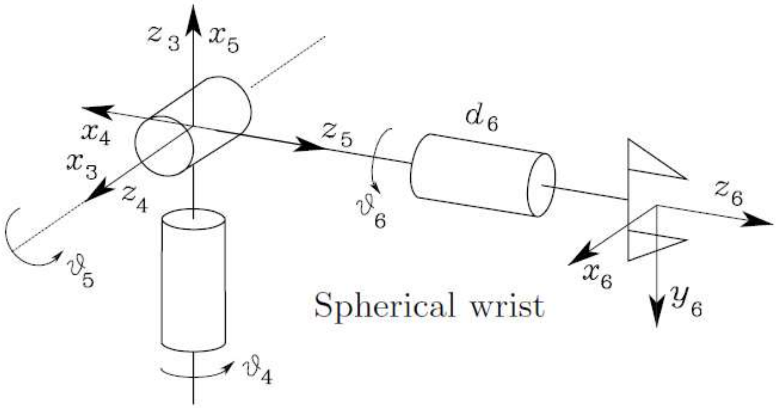
Recall Example 3: spherical wrist



To REMEMBER

a_i --> distance Zi and Zi-1 along xi

α_i -->angle Zi and Zi-1 around xi

d_i -->distance Xi and Xi-1 along Zi-1

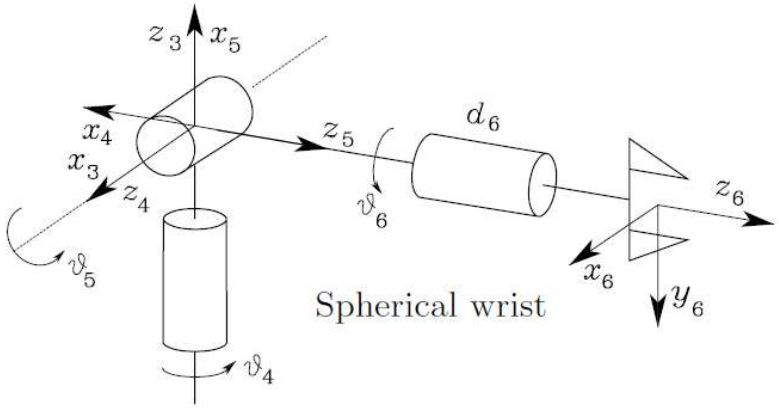
θ_i --> angle Xi and Xi-1 around Zi-1

DH parameters for the spherical wrist

Link	a_i	α_i	d_i	ϑ_i
4	0	$-\pi/2$	0	ϑ_4
5	0	$\pi/2$	0	ϑ_5
6	0	0	d_6	ϑ_6

Kinematics of Typical Manipulator Structures (D-H)

Recall Example 3: spherical wrist



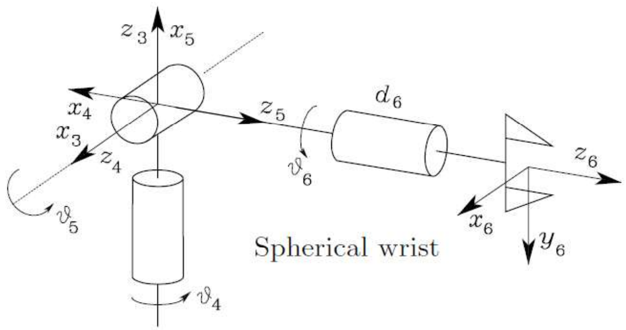
$$A_4^3(\vartheta_4) = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad A_5^4(\vartheta_5) = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_6^5(\vartheta_6) = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

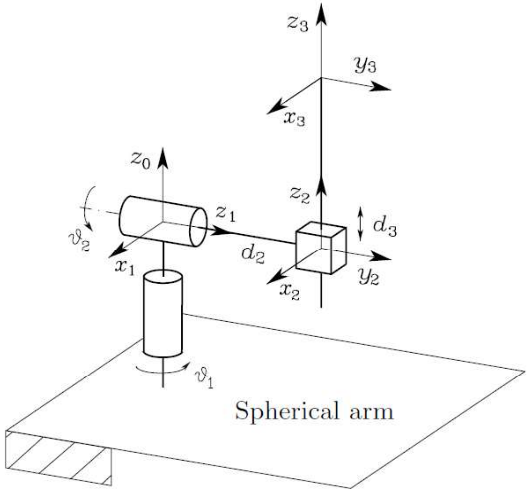
$$T_6^3(q) = A_4^3 A_5^4 A_6^5 = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 & c_4s_5d_6 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 & s_4s_5d_6 \\ -s_5c_6 & s_5s_6 & c_5 & c_5d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 8: Stanford Manipulator

$$T_6^3(q) = A_4^3 A_5^4 A_6^5 = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 & c_4s_5d_6 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 & s_4s_5d_6 \\ -s_5c_6 & s_5s_6 & c_5 & c_5d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

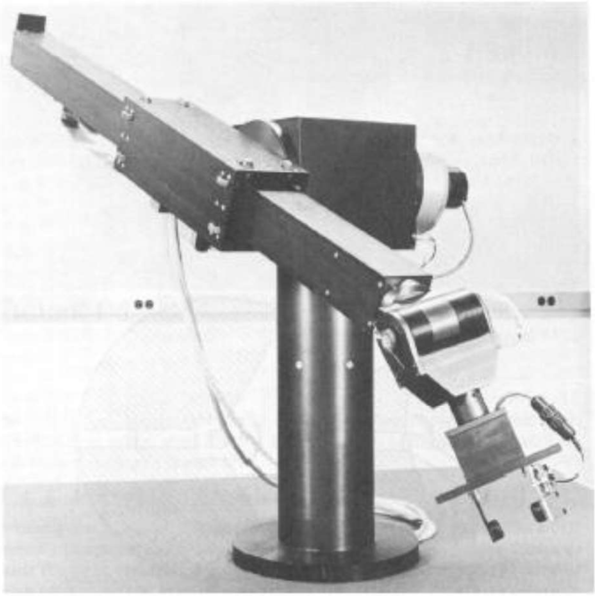


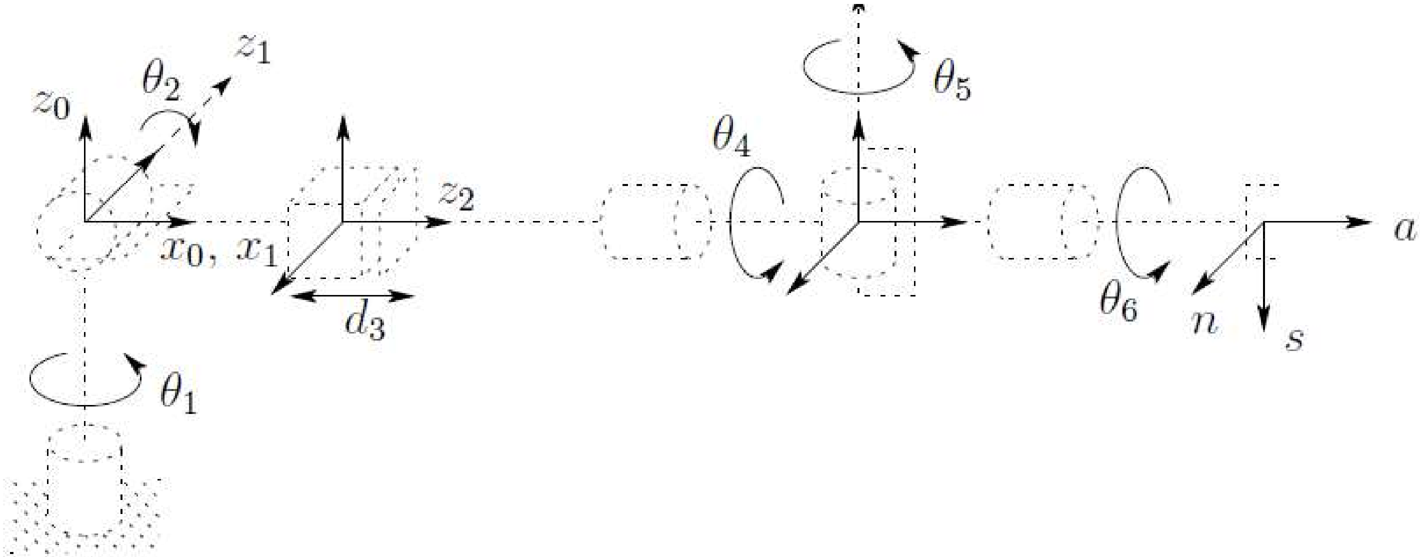
$$T_6^0 = T_3^0 T_6^3$$



$$T_3^0(q) = A_1^0 A_2^1 A_3^2 = \begin{bmatrix} c_1c_2 & -s_1 & c_1s_2 & c_1s_2d_3 - s_1d_2 \\ s_1c_2 & c_1 & s_1s_2 & s_1s_2d_3 + c_1d_2 \\ -s_2 & 0 & c_2 & c_2d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 9: Stanford Manipulator (entire structure)





This manipulator is an example of a spherical (RRP) manipulator with a spherical wrist. This manipulator has an offset in the shoulder joint that slightly complicates both the forward and inverse kinematics problems.

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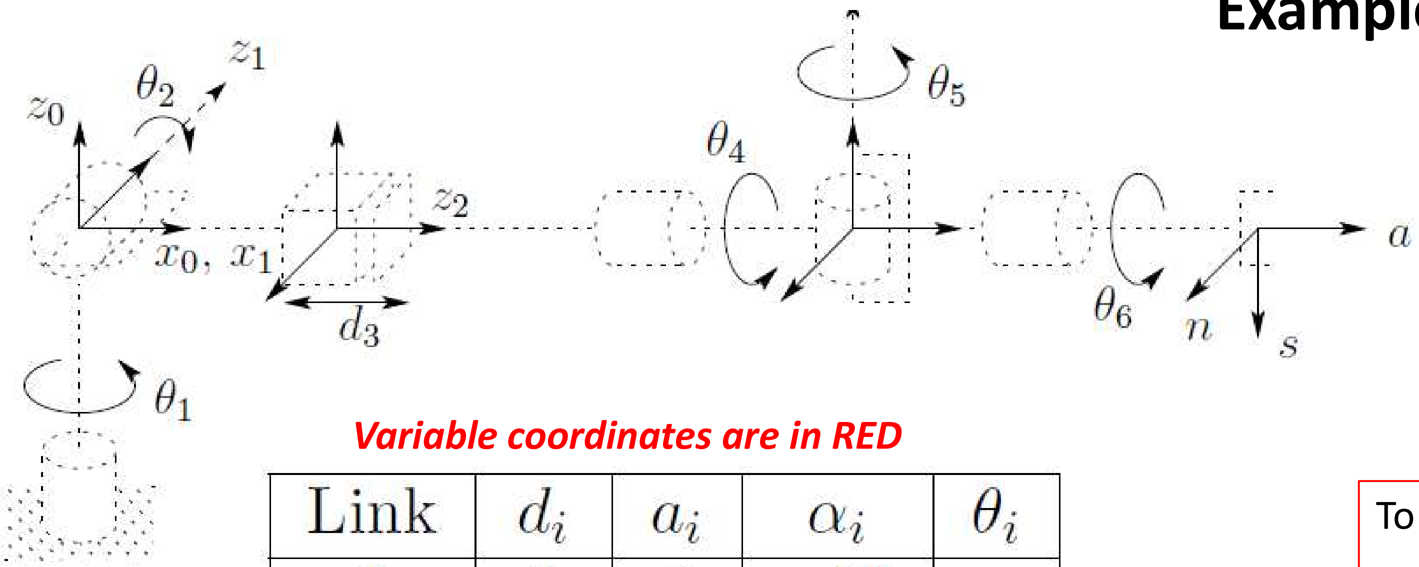
We first establish the joint coordinate frames using the D-H convention as shown below.

Example 9: Stanford Manipulator



Variable coordinates are in RED

Link	d_i	a_i	α_i	θ_i
1	0	0	-90	θ_1
2	d_2	0	$+90$	θ_2
3	d_3	0	0	0
4	0	0	-90	θ_4
5	0	0	$+90$	θ_5
6	d_6	0	0	θ_6

To REMEMBER

a_i --> distance Zi and Zi-1 along xi

α_i -->angle Zi and Zi-1 around xi

d_i -->distance Xi and Xi-1 along Zi-1

θ_i --> angle Xi and Xi-1 around Zi-1

Example 9: Stanford Manipulator

Link	d_i	a_i	α_i	θ_i
1	0	0	-90	θ_1
2	d_2	0	$+90$	θ_2
3	d_3	0	0	0
4	0	0	-90	θ_4
5	0	0	$+90$	θ_5
6	d_6	0	0	θ_6

It is straightforward to compute the matrices A_i as

$$A_1 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_5 = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_6 = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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Example 9: Stanford Manipulator

T_6^0 *is then given as*

$$T_6^0 = A_1 \cdots A_6 = \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation of the end-effector respect the frame x0 y0 z0

$$\begin{aligned}
r_{11} &= c_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] - d_2(s_4c_5c_6 + c_4s_6) \\
r_{21} &= s_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] + c_1(s_4c_5c_6 + c_4s_6) \\
r_{31} &= -s_2(c_4c_5c_6 - s_4s_6) - c_2s_5c_6 \\
r_{12} &= c_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] - s_1(-s_4c_5s_6 + c_4c_6) \\
r_{22} &= -s_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] + c_1(-s_4c_5s_6 + c_4c_6) \\
r_{32} &= s_2(c_4c_5s_6 + s_4c_6) + c_2s_5s_6 \\
r_{13} &= c_1(c_2c_4s_5 + s_2c_5) - s_1s_4s_5 \\
r_{23} &= s_1(c_2c_4s_5 + s_2c_5) + c_1s_4s_5 \\
r_{33} &= -s_2c_4s_5 + c_2c_5
\end{aligned}$$

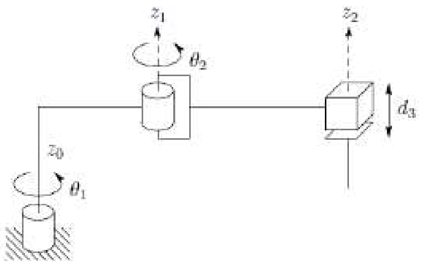
Coordinates of the end-effector respect to the base

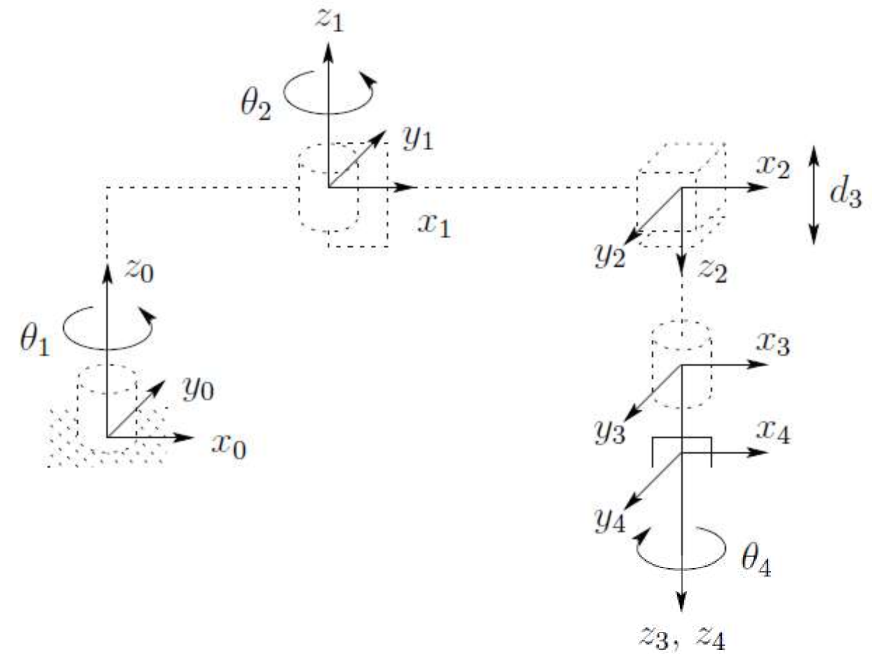
$$\begin{aligned}
d_x &= c_1s_2d_3 - s_1d_2 + +d_6(c_1c_2c_4s_5 + c_1c_5s_2 - s_1s_4s_5) \\
d_y &= s_1s_2d_3 + c_1d_2 + d_6(c_1s_4s_5 + c_2c_4s_1s_5 + c_5s_1s_2) \\
d_z &= c_2d_3 + d_6(c_2c_5 - c_4s_2s_5).
\end{aligned}$$

THE DENAVIT-HARTENBERG CONVENTION

EXAMPLE 10: SCARA Manipulator

As another example of the general procedure, consider the SCARA manipulator.
This manipulator, consists of an RRP arm and a one degree-of-freedom wrist, whose motion is a roll about the vertical axis.





EXAMPLE 10: SCARA Manipulator

Variable coordinates are in RED

Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	θ_1
2	a_2	180	0	θ_2
3	0	0	d_3	0
4	0	0	d_4	θ_4

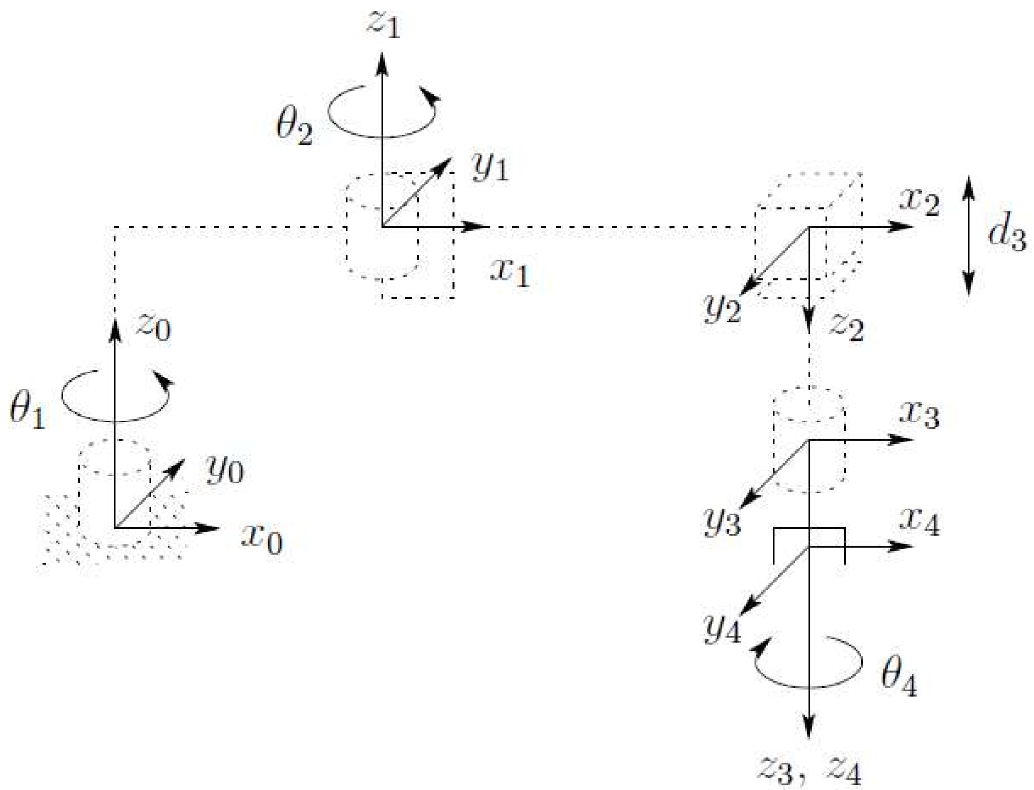
To REMEMBER

a_i --> distance Zi and Zi-1 along xi

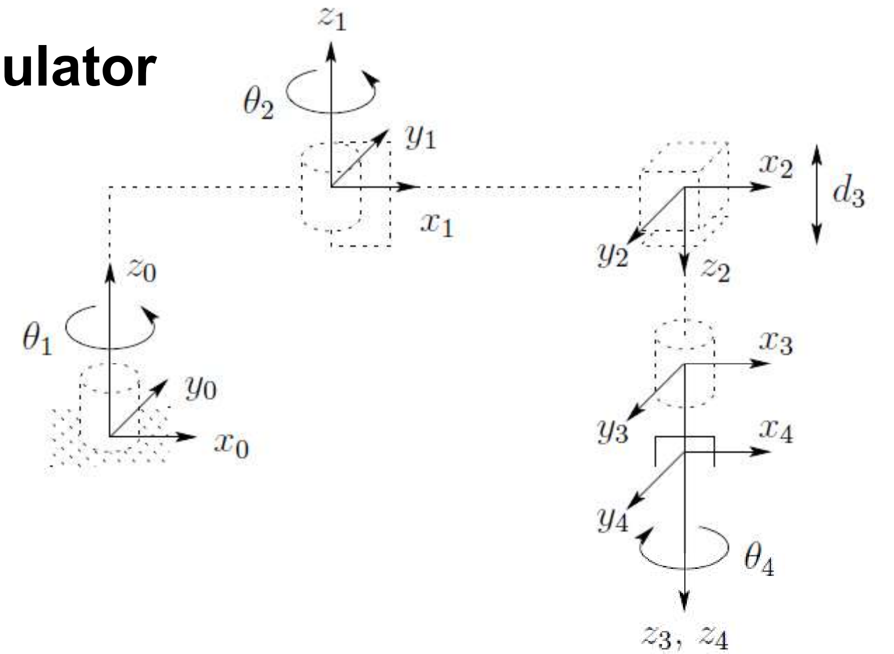
α_i -->angle Zi and Zi-1 around xi

d_i -->distance Xi and Xi-1 along Zi-1

θ_i --> angle Xi and Xi-1 around Zi-1



EXAMPLE 10: SCARA Manipulator



$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & a_1c_1 \\ s_1 & c_1 & 0 & a_1s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} c_2 & s_2 & 0 & a_2c_2 \\ s_2 & -c_2 & 0 & a_2s_2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

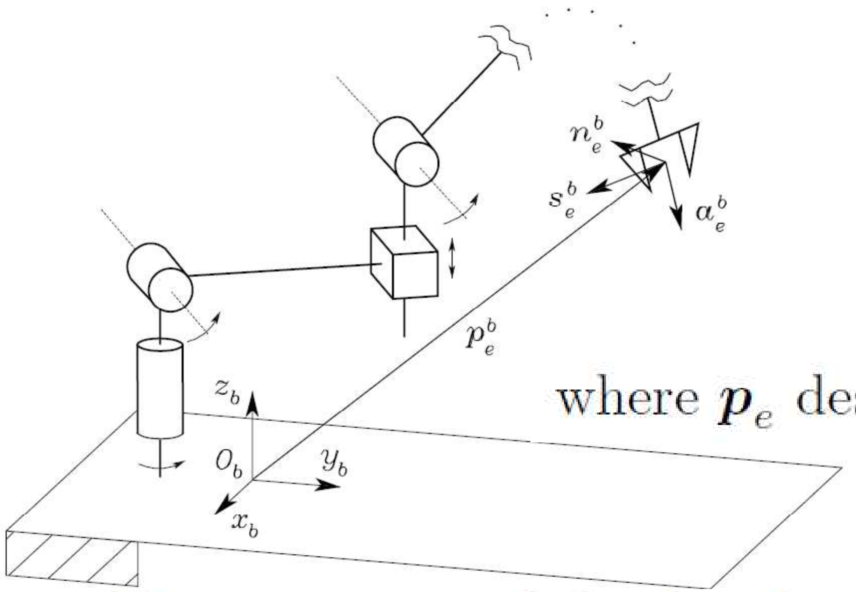
$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} c_4 & -s_4 & 0 & 0 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The forward kinematic equations are therefore given by

$$T_4^0 = A_1 \cdots A_4 = \begin{bmatrix} c_{12}c_4 + s_{12}s_4 & -c_{12}s_4 + s_{12}c_4 & 0 & a_1c_1 + a_2c_{12} \\ s_{12}c_4 - c_{12}s_4 & -s_{12}s_4 - c_{12}c_4 & 0 & a_1s_1 + a_2s_{12} \\ 0 & 0 & -1 & -d_3 - d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Joint Space and Operational Space (kinematic redundancy)



$$\boldsymbol{x}_e = \begin{bmatrix} \boldsymbol{p}_e \\ \boldsymbol{\phi}_e \end{bmatrix}$$

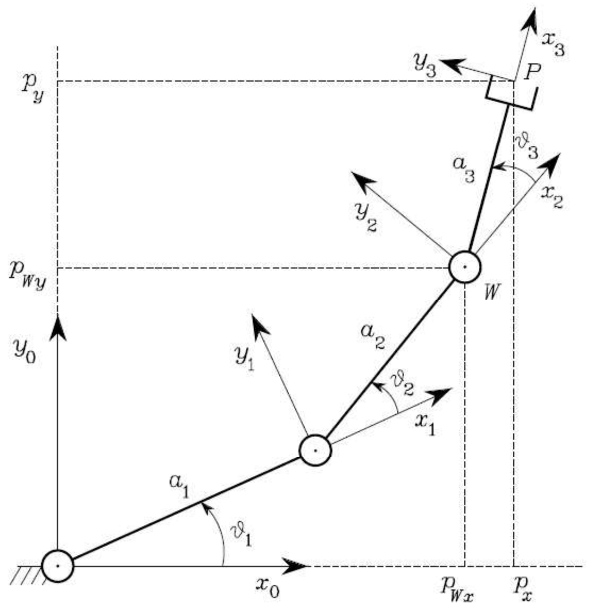
where $\boldsymbol{p}_e$ describes the end-effector position and $\boldsymbol{\phi}_e$ its orientation.

The vector $\boldsymbol{x}_e$ is defined in the space in which the manipulator task is specified; hence, this space is typically called *operational space*.

joint space (configuration space) denotes the space in which the $(n \times 1)$ vector of joint variables

$$\boldsymbol{q} = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$$

Joint Space and Operational Space: Example



$$
\boldsymbol{T}_3^0(\boldsymbol{q}) = \boldsymbol{A}_1^0 \boldsymbol{A}_2^1 \boldsymbol{A}_3^2 = \begin{bmatrix} c_{123} & -s_{123} & 0 & a_1 c_1 + a_2 c_{12} + a_3 c_{123} \\ s_{123} & c_{123} & 0 & a_1 s_1 + a_2 s_{12} + a_3 s_{123} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

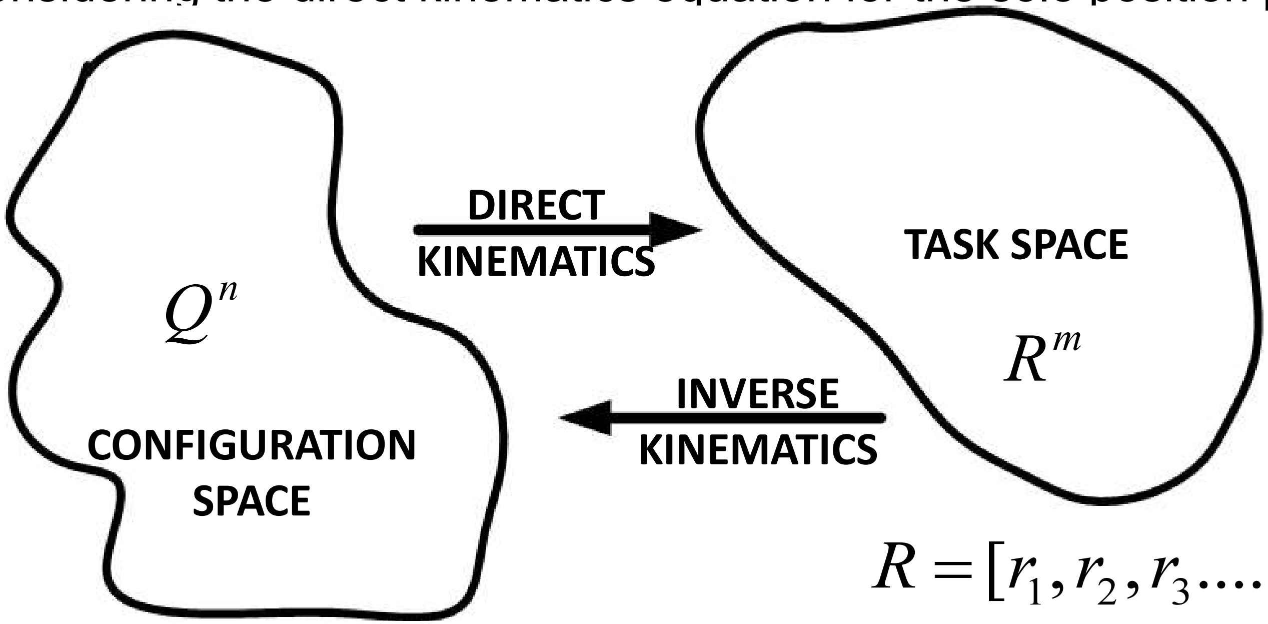
$$
\boldsymbol{x}_e = \begin{bmatrix} p_x \\ p_y \\ \phi \end{bmatrix} = \boldsymbol{k}(\boldsymbol{q}) = \begin{bmatrix} a_1 c_1 + a_2 c_{12} + a_3 c_{123} \\ a_1 s_1 + a_2 s_{12} + a_3 s_{123} \\ \vartheta_1 + \vartheta_2 + \vartheta_3 \end{bmatrix}
$$

Kinematic Redundancy $(n > m)$

A manipulator is termed *kinematically redundant* when it has a number of DOFs which is greater than the number of variables that are necessary to describe a task.

For an *n*-DOF manipulator, the reachable workspace is the geometric locus of the points that can be achieved by considering the direct kinematics equation for the sole position part

$$p_e = p_e(q)$$



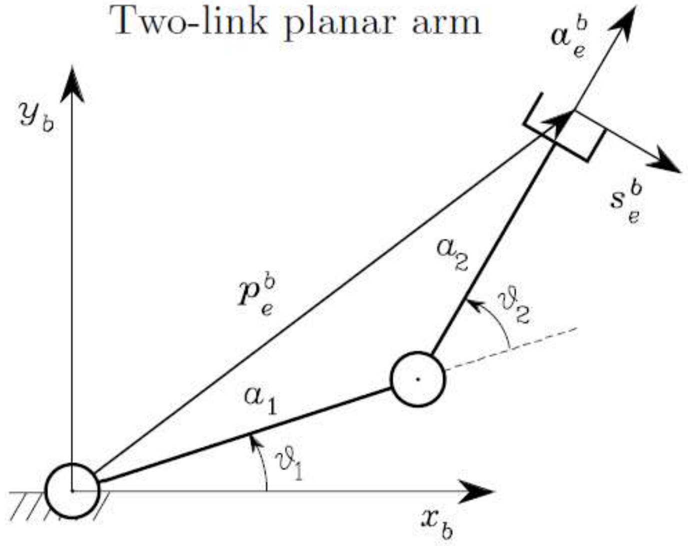
$$R = [r_1, r_2, r_3 \ldots\ldots r_m]$$

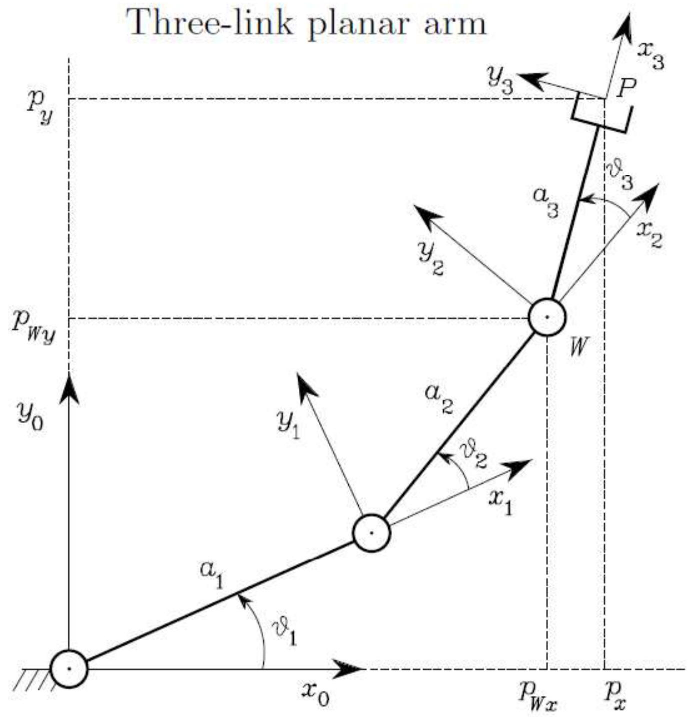
$$Q = [q_1, q_2, q_3 \ldots\ldots q_n]$$

Kinematic Redundancy

a manipulator is intrinsically redundant when the dimension of the operational space is smaller than the dimension of the joint space ($m < n$).

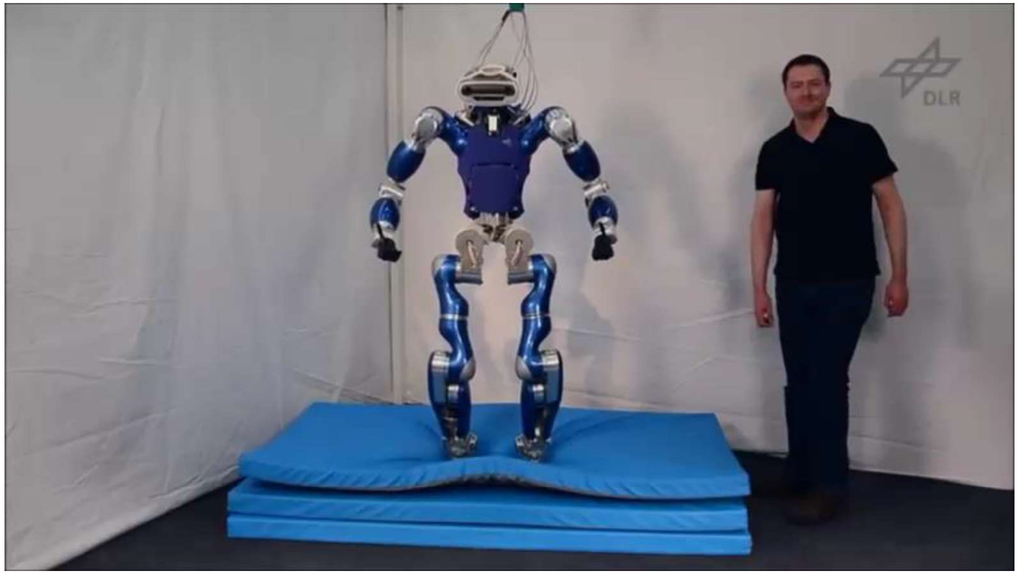
($m = n$).





($m < n$).

Example of kinematic redundancy



The end!