INVERSE KINEMATICS
Analytic Solution

Classes online

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https://www.youtube.com/playlist?list=PLAQopGWlIcyaqDBW1zSKx7lHfVcOmWSWt

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Functional Electrical Stimulation
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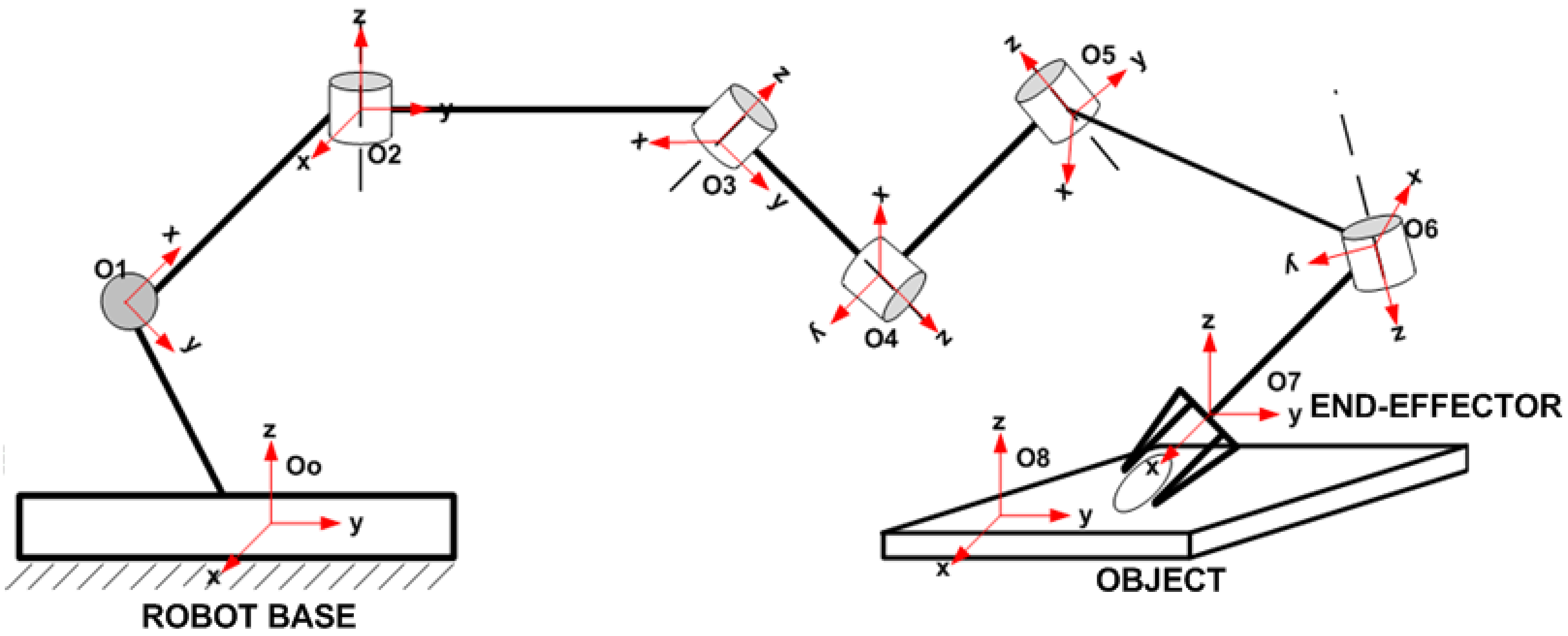
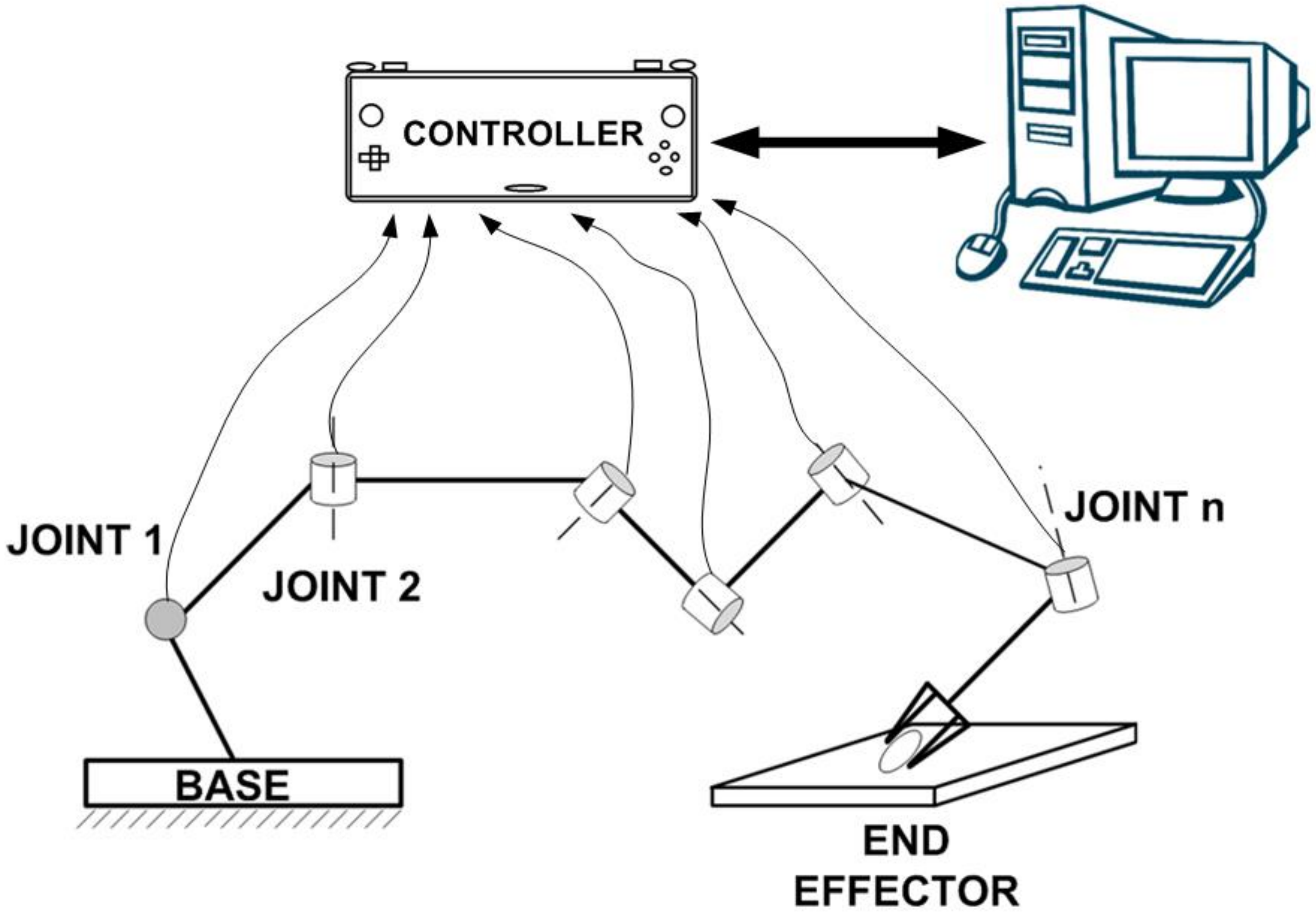
https://www.youtube.com/watch?v=jVu-Hijns70&list=PLggLP4f-rq02vX0OQQ5vrCxbJrzamYDfx

Robot Kinematics

First problem in programming robots is to describe the position of the «**end-effector**» in relation to a fixed frame usually called «**base**»

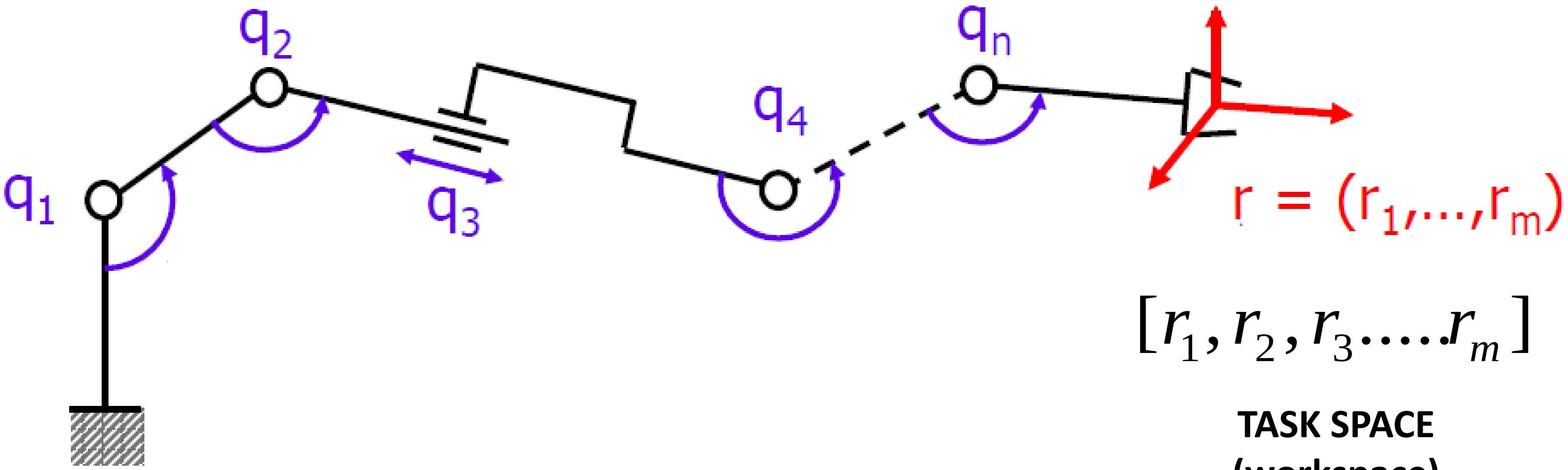


kinematics



Robot joints are equipped with sensors (encoders or resolvers) feeding back their ***rotation*** to the central CPU

kinematics



$$[r_1, r_2, r_3 \ldots . r_m]$$

TASK SPACE
(workspace)

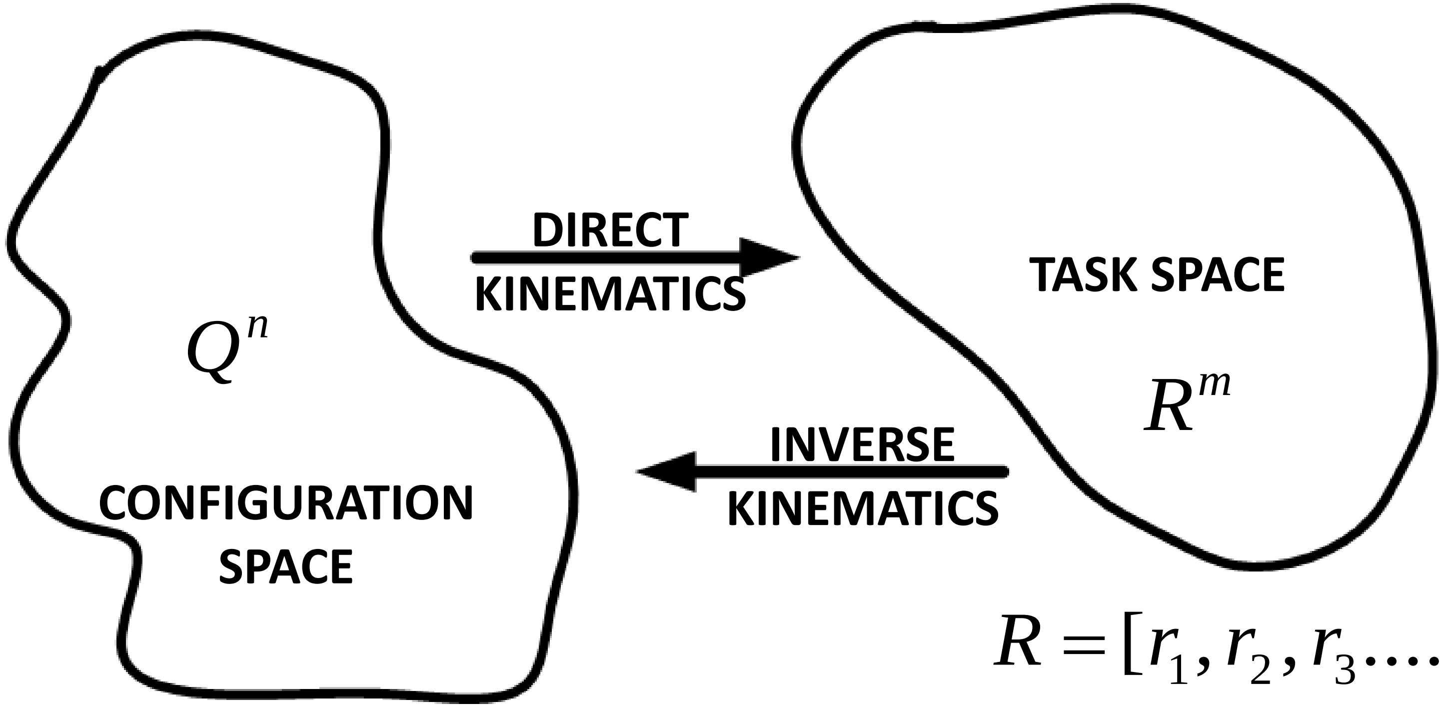
$$[q_1, q_2, q_3 \ldots . q_n]$$

CONFIGURATION
SPACE
(joints space)

How to relate the two SPACES?

$$[q_1, q_2, q_3 \ldots . q_n] \Longleftrightarrow [r_1, r_2, r_3 \ldots . r_m]$$

Kinematics



$$R = [r_1, r_2, r_3 \ldots . r_m]$$

$$Q = [q_1, q_2, q_3 \ldots . q_n]$$

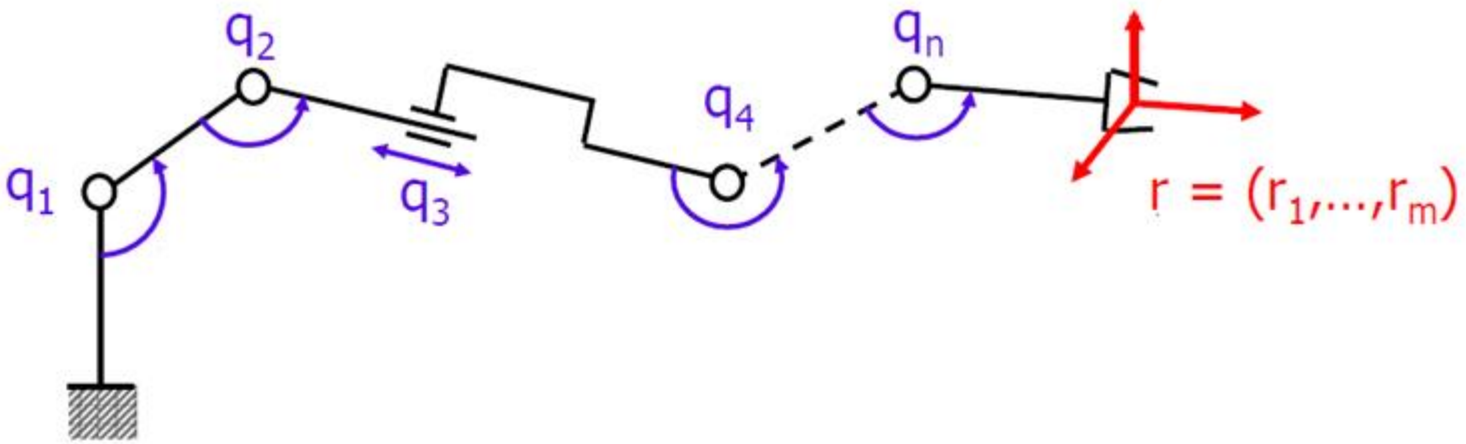
- The dimension of the *configuration space* must be **larger or equal** to the dimension of the *task space* $(n \geq m)$

- To ensure the existence of Kinematics solutions.

Inverse Kinematics

The process of finding the joint angles that realizes a given (desired) position/orientation of the end-effector is known as *inverse kinematics*.

$$(q_1, q_2, q_3....., q_n) = G(r_1, r_2, r_3.....r_m)$$



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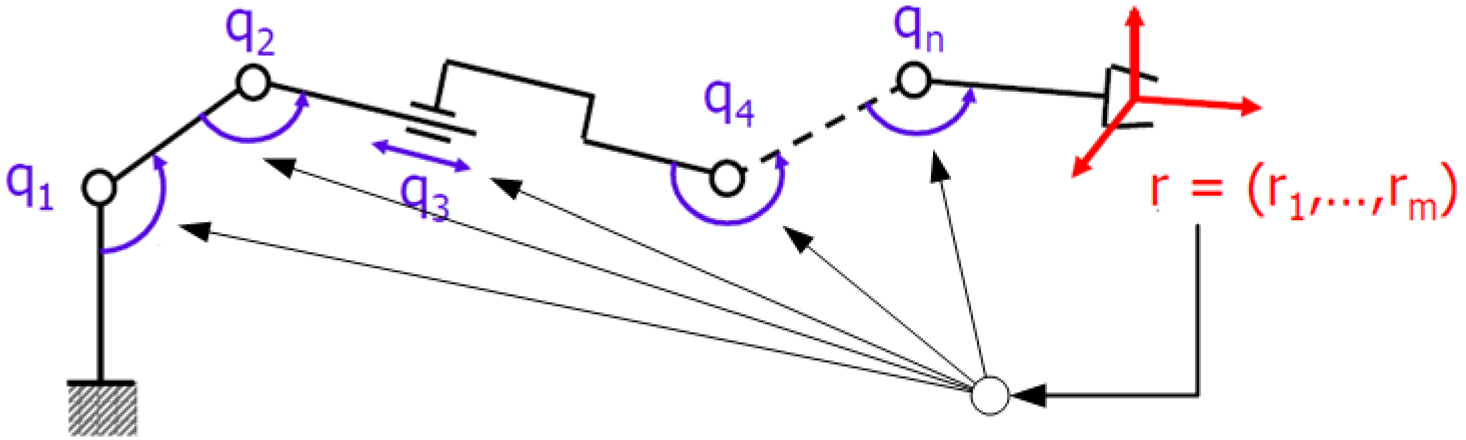
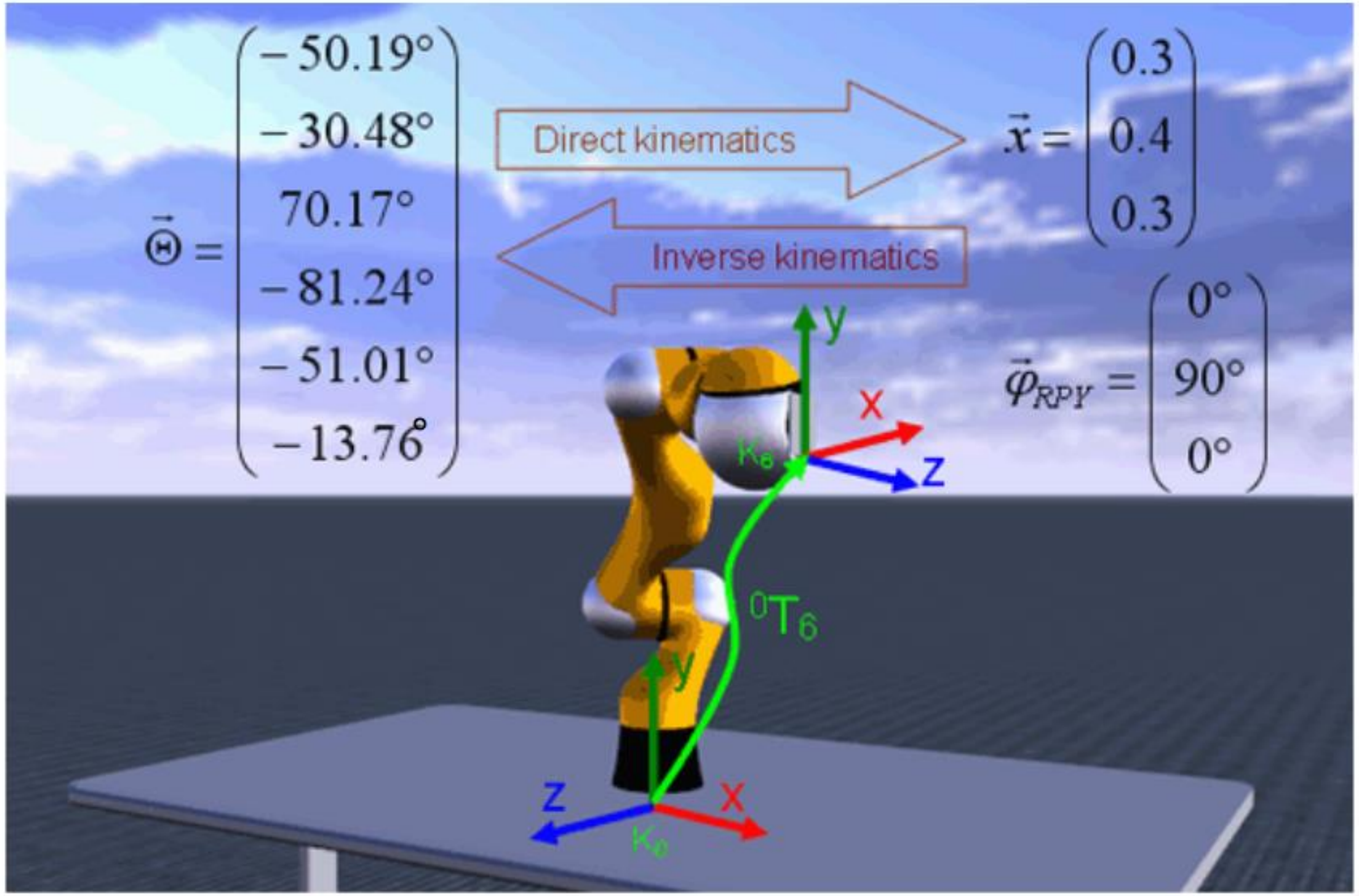
inverse Kinematics

The process of finding the joint angles that realizes a given position/orientation of the end-effector is known as *inverse kinematics*.

$$(q_1, q_2, q_3 \ldots\ldots, q_n) = G(r_1, r_2, r_3 \ldots\ldots r_m)$$

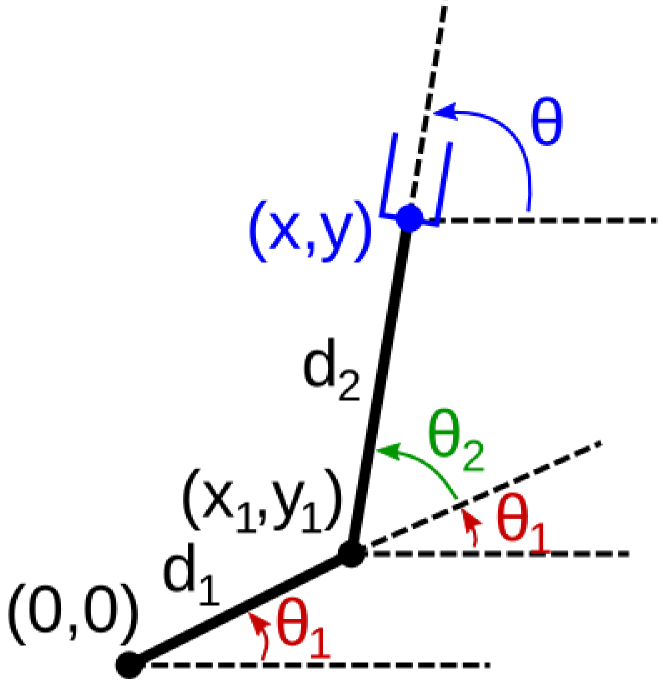


Inverse kinematics
what are we looking for?



direct kinematics is always unique;
how about inverse kinematics for this 6R robot?

Example: inverse Kinematics



TASK:
to place the gripper at a desired position:

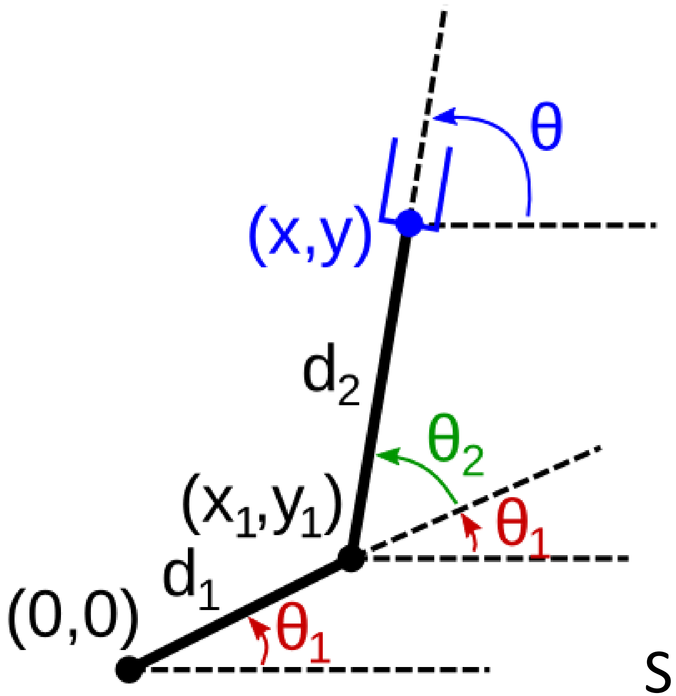
$$\boldsymbol{p}_{\text{des}} := (x_{\text{des}}, y_{\text{des}})$$

Finding the appropriate ***joint angles*** that achieve this position it constitutes the ***inverse kinematics*** problem:

$$\boldsymbol{q}^* := (\theta_1^*, \theta_2^*)$$

Unknown → (θ_1^*, θ_2^*)

Example: inverse Kinematics



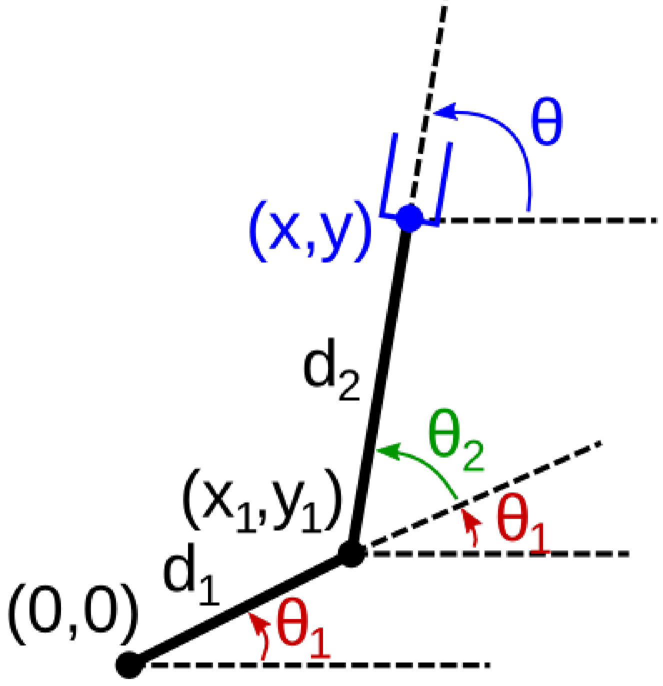
The forward kinematic provided:

$$\begin{cases} x_{\text{des}} &= d_1\cos(\theta_1^*) + d_2\cos(\theta_1^* + \theta_2^*) \\ y_{\text{des}} &= d_1\sin(\theta_1^*) + d_2\sin(\theta_1^* + \theta_2^*) \end{cases}$$

Squaring both sides of equation and summing them up:

$$\begin{aligned} x_{\text{des}}^2 + y_{\text{des}}^2 &= d_1^2 + d_2^2 + 2d_1d_2\left(\cos(\theta_1^*)\cos(\theta_1^* + \theta_2^*) + \sin(\theta_1^*)\sin(\theta_1^* + \theta_2^*)\right) \\ &= d_1^2 + d_2^2 + 2d_1d_2\cos(\theta_2^*). \end{aligned}$$

Example: inverse Kinematics



The forward kinematic provided:

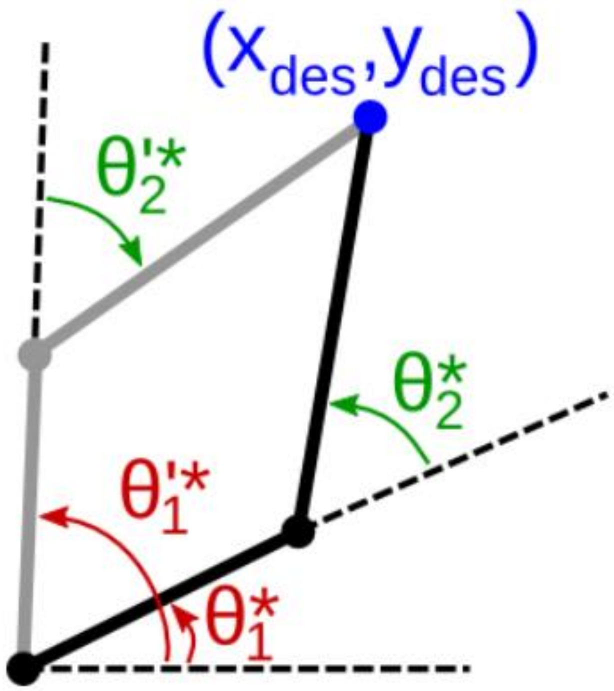
$$x_{\text{des}}^2 + y_{\text{des}}^2 = d_1^2 + d_2^2 + 2d_1 d_2 \cos(\theta_2^*).$$

$$\cos(\theta_2^*) = \frac{x_{\text{des}}^2 + y_{\text{des}}^2 - d_1^2 - d_2^2}{2d_1 d_2}.$$

$$\theta_2^* = \pm \arccos\left(\frac{x_{\text{des}}^2 + y_{\text{des}}^2 - d_1^2 - d_2^2}{2d_1 d_2}\right)$$

There are two values of the angle. Why?

Example: inverse Kinematics



Next, after some calculations, one can find the expression for the two angles:

$$\theta_2^* = \pm \arccos\left(\frac{x_{\text{des}}^2 + y_{\text{des}}^2 - d_1^2 - d_2^2}{2 d_1 d_2}\right)$$

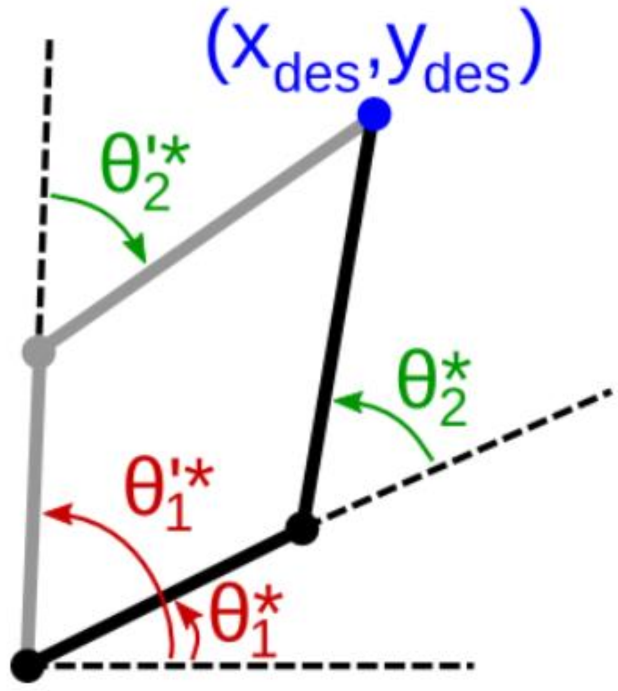
$$\theta_1^* = \arctan 2(y_{\text{des}}, x_{\text{des}}) - \arctan 2(k_2, k_1),$$

where

$$k_1 := d_1 + d_2 \cos(\theta_2^*) \quad \text{and} \quad k_2 := d_2 \sin(\theta_2^*).$$

Appendix for compete calculation.

EXAMPLE: INVERSE KINEMATICS

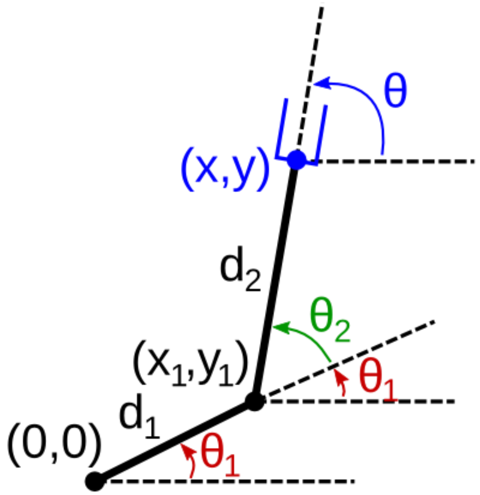


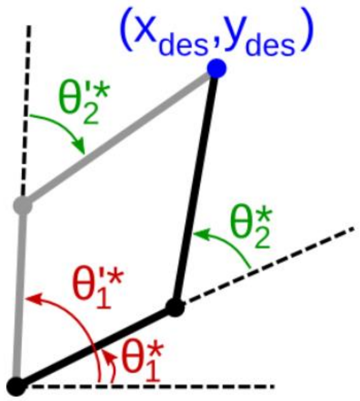
The above derivations raise the following remarks:

- Inverse kinematics calculations are in general much more difficult than forward kinematics calculations;
- While a configuration $\boldsymbol{q}$ always yields *one* forward kinematics solution $\boldsymbol{p}$, a given desired end-effector position $\boldsymbol{p}_{\text{des}}$ may correspond to zero, one, or multiple possible IK solutions $\boldsymbol{q}^*$.

Redundancy (definition)

<u>Redundancy</u> arises when there are *<u>multiple Inverse Kinematics</u>* solutions for a given desired task value.





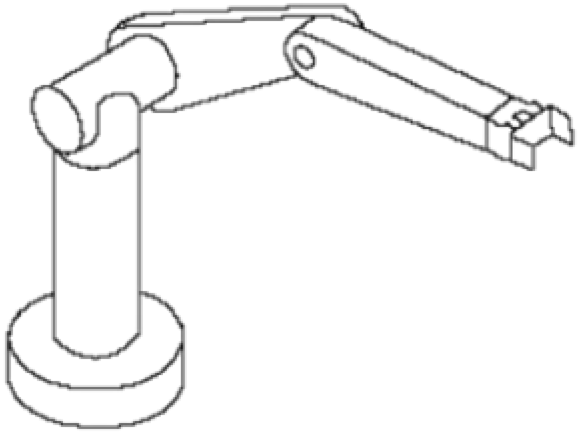
Task Space	Configuration Space
$(x_{\text{des}}, y_{\text{des}})$	(θ_1^*, θ_2^*) $(\theta'^*_1, \theta'^*_2)$

Inverse Kinematics Caveats

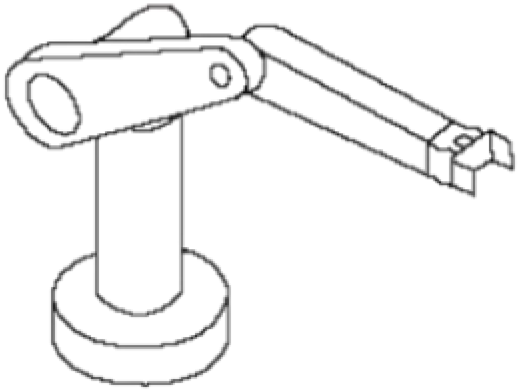
1. May or may not have a solution
2. Even if solution exists, may or may not be unique
3. Because forward kinematics is generally nonlinear, solutions can be hard to obtain even if they exist

Example: pretend you are a PUMA arm – how many solutions?

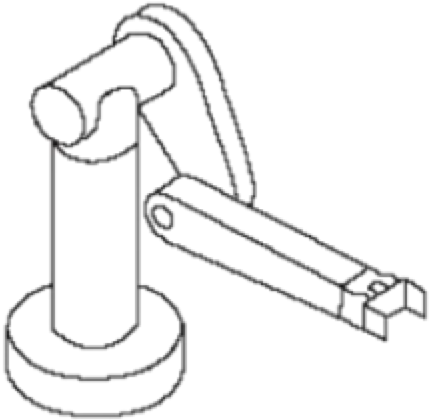
Multiple solutions



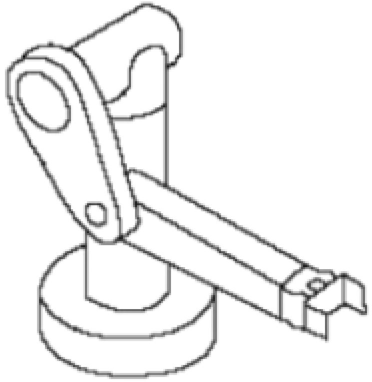
Left Arm Elbow Up



Right Arm Elbow Up



Left Arm Elbow Down



Right Arm Elbow Down

Inverse kinematics problem

- "given a desired end-effector pose (position + orientation), find the values of the joint variables that will realize it"
- a synthesis problem, with input data in the form
  - $T = \begin{bmatrix} R & p \\ 000 & 1 \end{bmatrix} = {}^0A_n(q)$
  - $r = \begin{bmatrix} p \\ \phi \end{bmatrix} = f_r(q)$, or for any other task vector

classical formulation:
inverse kinematics for a given end-effector pose

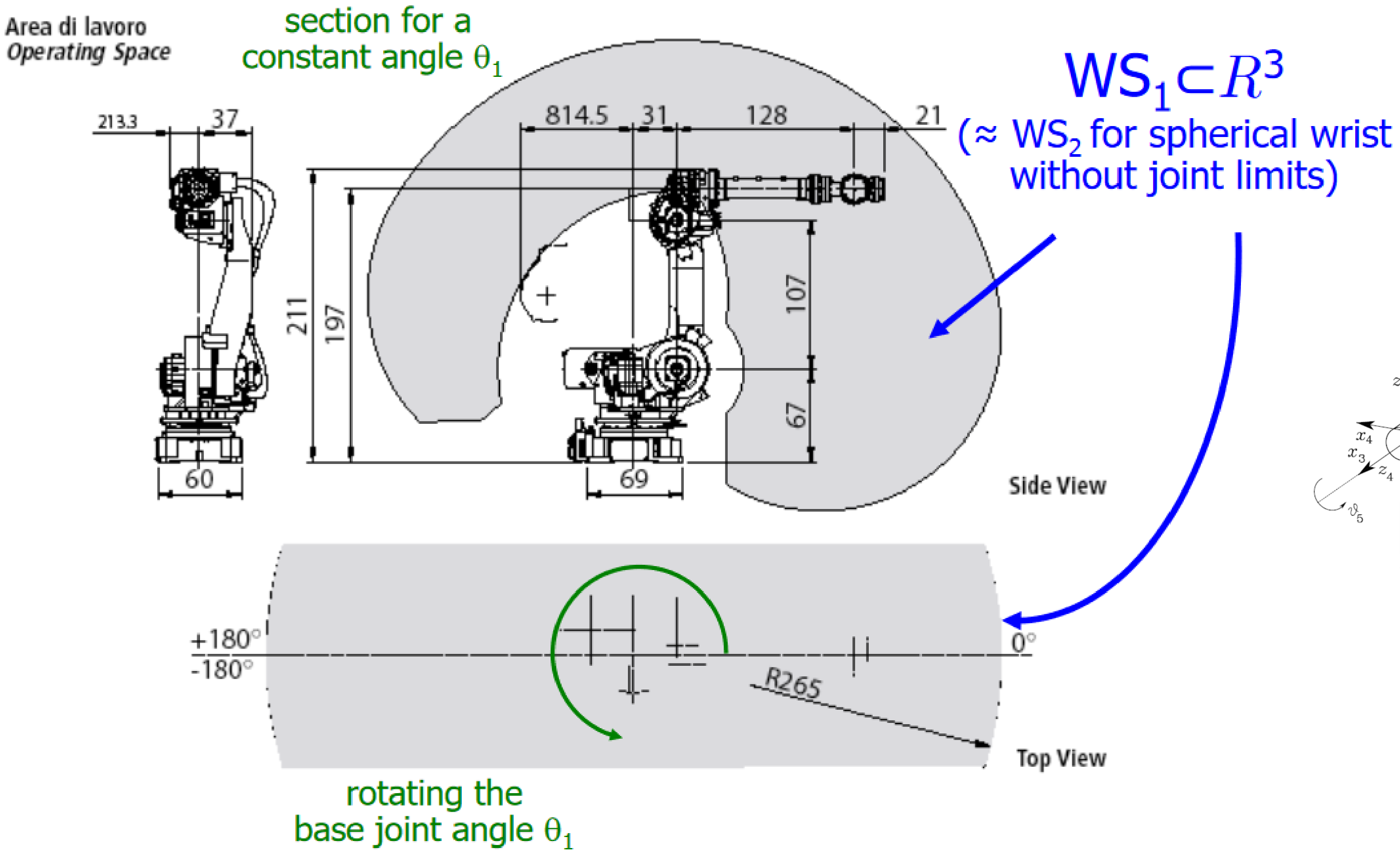
generalized formulation:
inverse kinematics for a given value of task variables

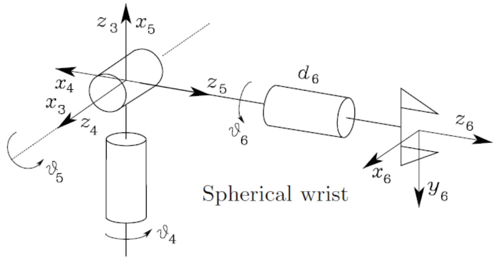
- a typical nonlinear problem
  - existence of a solution (workspace definition)
  - uniqueness/multiplicity of solutions ($r \in \mathcal{R}^m$, $q \in \mathcal{R}^n$)
  - solution methods

Solvability and robot workspace
(for tasks related to a desired end-effector Cartesian pose)

- primary workspace WS_1: set of all positions p that can be reached with at least one orientation (ϕ or R)
  - out of WS_1 there is no solution to the problem
  - when $p \in WS_1$, there is a suitable ϕ (or R) for which a solution exists
- secondary (or *dexterous*) workspace WS_2: set of positions p that can be reached with any orientation (among those feasible for the robot direct kinematics)
  - when $p \in WS_2$, there exists a solution for any feasible ϕ (or R)

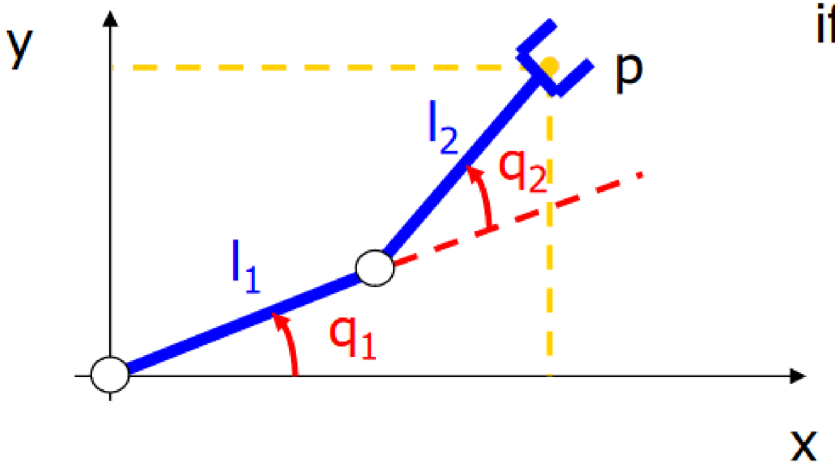
Workspace of Fanuc R-2000i/165F

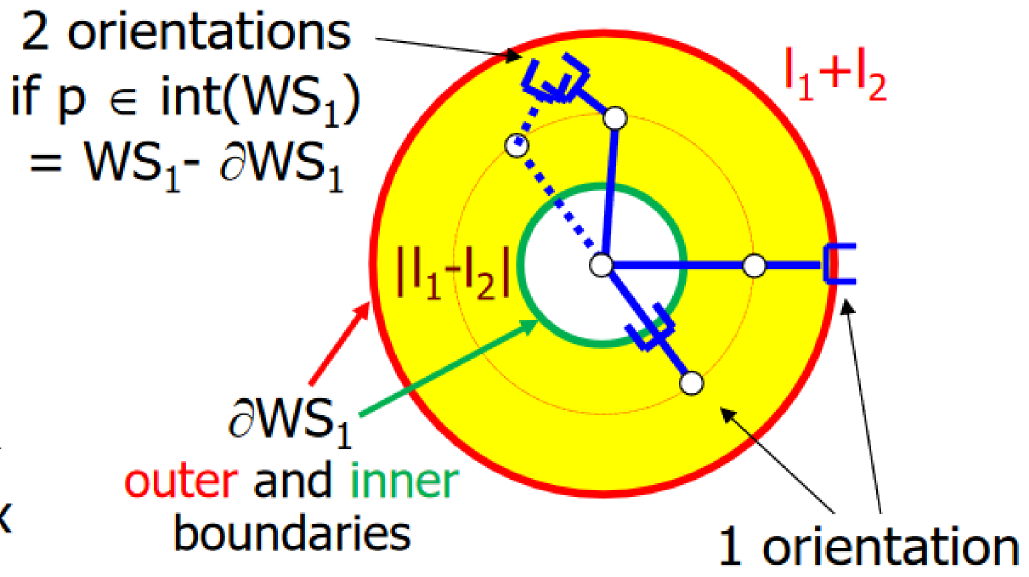




If you assume that the spherical wrist is with no rotational limits primary and secondary workspace are coincident, Because for all the position that I reach I can assume any orientation.

Workspace of planar 2R arm



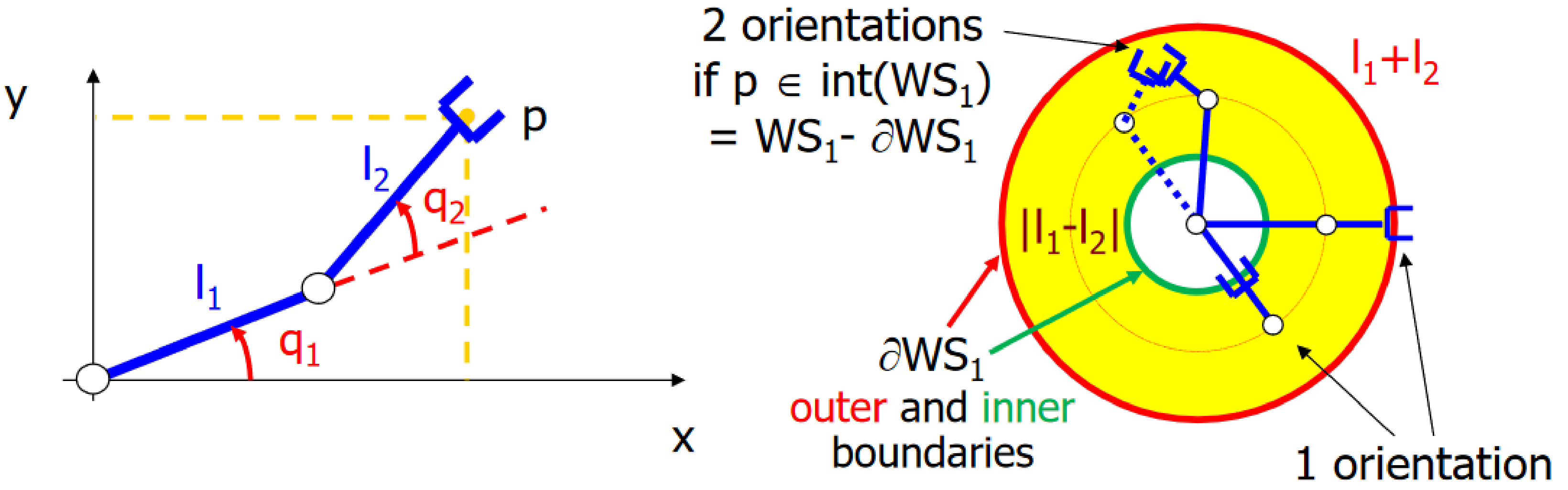


- if $l_1 \neq l_2$
  - $WS_1 = \{p \in \mathcal{R}^2: |l_1-l_2| \leq \|p\| \leq l_1+l_2\} \subset \mathcal{R}^2$
  - $WS_2 = \varnothing$
- if $l_1 = l_2 = \ell$
  - $WS_1 = \{p \in \mathcal{R}^2: \|p\| \leq 2\ell\} \subset \mathcal{R}^2$
  - $WS_2 = \{p = 0\}$ (infinite number of feasible orientations at the origin)

- E-E positioning (m=2) of a planar 2R robot arm
  - 2 regular solutions in int(WS_1)
  - 1 solution on ∂WS_1
  - for $l_1 = l_2$: ∞ solutions in WS_2

  } *singular* solutions

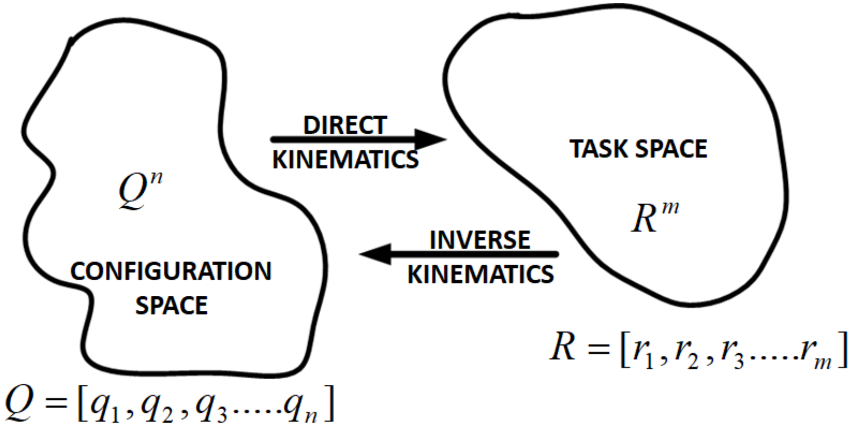
Workspace of planar 2R arm



When we will study differential kinematics using the Jacobian matrix we will see that in Singular configuration the jacobian becomes singular (determinant is zero).

Possible situations

- if m = n
  - ∄ solutions
  - a finite number of solutions (regular/generic case)
  - "degenerate" solutions: infinite or finite set, but anyway different in number from the generic case (singularity)
- if m < n (robot is redundant for the kinematic task)
  - ∄ solutions
  - ∞^{n-m} solutions (regular/generic case)
  - a finite or infinite number of singular solutions



In the case of the Kuka Light arm:
m=6
n=7
$\infty^{n-m} = \infty^1$

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How to compute the inverse kinematics?

ANALYTICAL solution (in closed form) ⇄ **NUMERICAL solution** (in iterative form)

ANALYTICAL solution (in closed form)

- preferred, if it can be found*
- use ad-hoc geometric inspection
- algebraic methods (solution of polynomial equations)
- systematic ways for generating a reduced set of equations to be solved

NUMERICAL solution (in iterative form)

- certainly needed if n>m (redundant case), or at/close to singularities
- slower, but easier to be set up
- in its basic form, it uses the (analytical) Jacobian matrix of the direct kinematics map

$$J_r(q) = \frac{\partial f_r(q)}{\partial q}$$

- Newton method, Gradient method, and so on…

$r = \begin{bmatrix} p \\ \phi \end{bmatrix} = f_r(q)$, or for any other task vector

generalized formulation: inverse kinematics for a given value of task variables

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Inverse Kinematics
Analytical Solution

the inverse problem of finding the joint variables in terms of the **end-effector position** and **orientation** it is, in general, more difficult than the forward kinematics problem.

To do list

- we begin by formulating the general inverse kinematics problem.
- we describe the principle of **kinematic decoupling** and how it can be used to simplify the inverse kinematics of most modern manipulators.
- **Using kinematic decoupling**, we can consider the position and orientation problems independently.
- We describe a geometric approach for solving the positioning problem, while we exploit the Euler angle parameterization to solve the orientation problem.

The General Inverse Kinematics Problem

The general problem of inverse kinematics can be stated as follows. Given a 4×4 homogeneous transformation

$$H = \begin{bmatrix} R & o \\ 0 & 1 \end{bmatrix} \in SE(3)$$

with $R \in SO(3)$, find (one or all) solutions of the equation

$$T_n^0(q_1, \ldots, q_n) = H \qquad \text{where} \qquad T_n^0(q_1, \ldots, q_n) = A_1(q_1) \cdots A_n(q_n).$$

Here, H represents the desired position and orientation of the end-effector, and our task is to find the values for the joint variables $q_1, \ldots, q_n$ so that $T_n^0(q_1, \ldots, q_n) = H$.

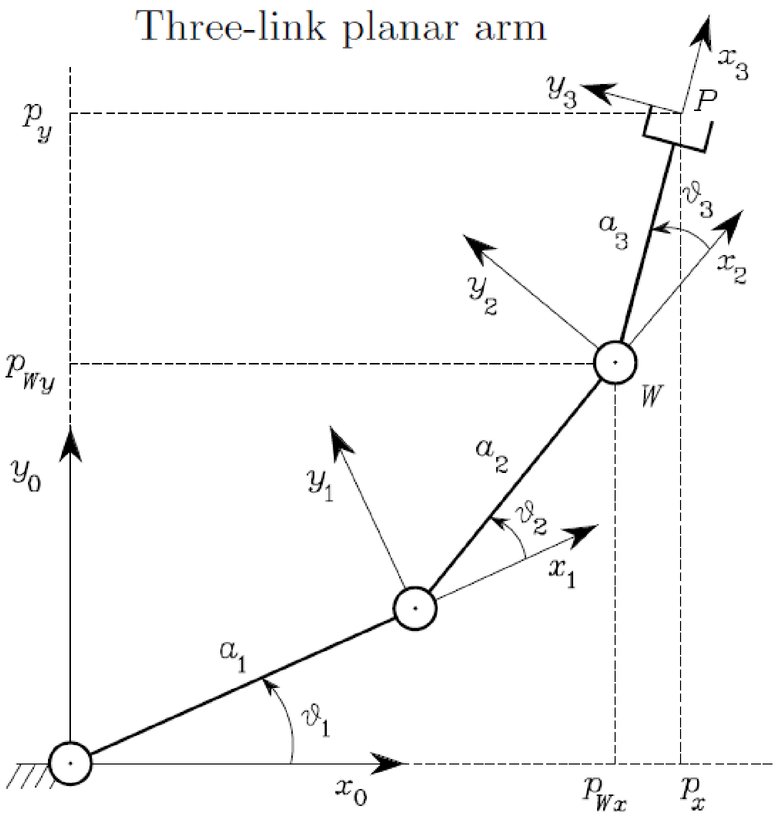
Inverse Kinematics Problem

The *inverse kinematics problem* consists of the determination of the joint variables corresponding to a given end-effector position and orientation.

On the other hand, the inverse kinematics problem is much more complex for the following reasons:

- The equations to solve are in general nonlinear, and thus it is not always possible to find a *closed-form solution*.
- *Multiple solutions* may exist.
- *Infinite solutions* may exist, e.g., in the case of a kinematically redundant manipulator.
- There might be no *admissible* solutions, in view of the manipulator kinematic structure.

Solution of Three-link Planar Arm



Position of point P

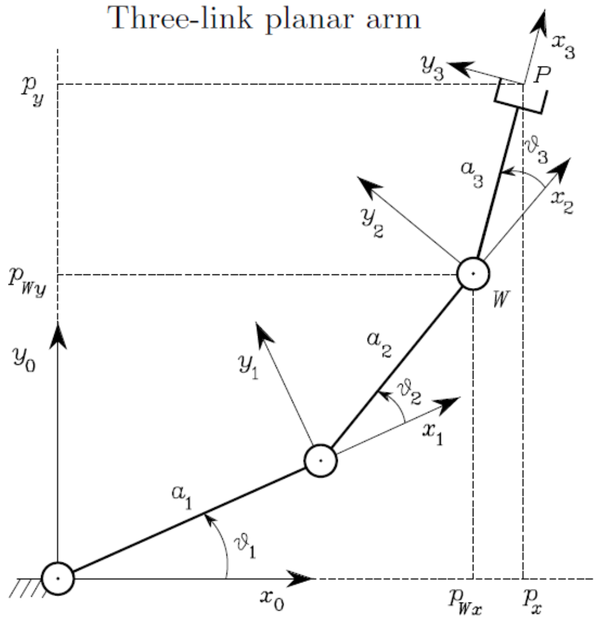
Find the joint variables ϑ_1, ϑ_2, ϑ_3 corresponding to a given end-effector position and orientation.

Remember the kinematic equation:

$$\boldsymbol{x}_e = \begin{bmatrix} p_x \\ p_y \\ \phi \end{bmatrix} = \boldsymbol{k}(\boldsymbol{q}) = \begin{bmatrix} a_1 c_1 + a_2 c_{12} + a_3 c_{123} \\ a_1 s_1 + a_2 s_{12} + a_3 s_{123} \\ \vartheta_1 + \vartheta_2 + \vartheta_3 \end{bmatrix}$$

$$\phi = \vartheta_1 + \vartheta_2 + \vartheta_3$$

Solution of Three-link Planar Arm



$$p_{Wx} = p_x - a_3 c_\phi = a_1 c_1 + a_2 c_{12}$$
$$p_{Wy} = p_y - a_3 s_\phi = a_1 s_1 + a_2 s_{12} \qquad (1)$$

Squaring and summing $\quad p_{Wx}^2 + p_{Wy}^2 = a_1^2 + a_2^2 + 2a_1 a_2 c_2$

$$c_2 = \frac{p_{Wx}^2 + p_{Wy}^2 - a_1^2 - a_2^2}{2a_1 a_2}.$$

Hence, the angle ϑ_2 can be computed as $\quad s_2 = \pm\sqrt{1 - c_2^2}, \quad \vartheta_2 = \text{Atan2}(s_2, c_2).$

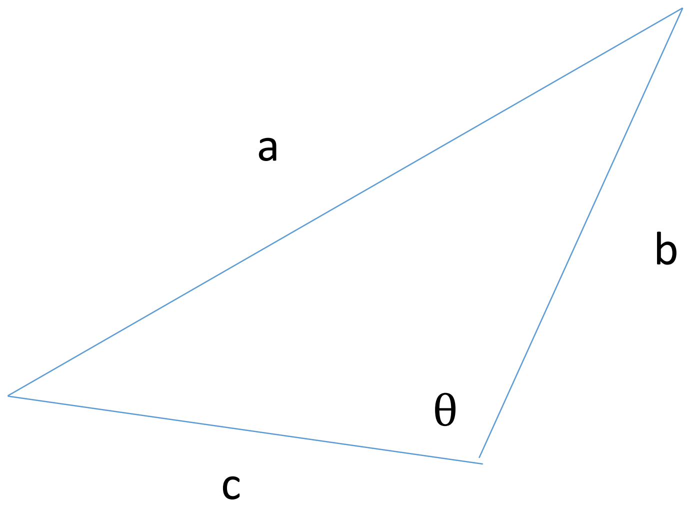
Substituting ϑ_2 into the (1) yields an algebraic system of two equations in the two unknowns s_1 and c_1, whose solution is

$$s_1 = \frac{(a_1 + a_2 c_2) p_{Wy} - a_2 s_2 p_{Wx}}{p_{Wx}^2 + p_{Wy}^2}$$
$$c_1 = \frac{(a_1 + a_2 c_2) p_{Wx} + a_2 s_2 p_{Wy}}{p_{Wx}^2 + p_{Wy}^2}.$$

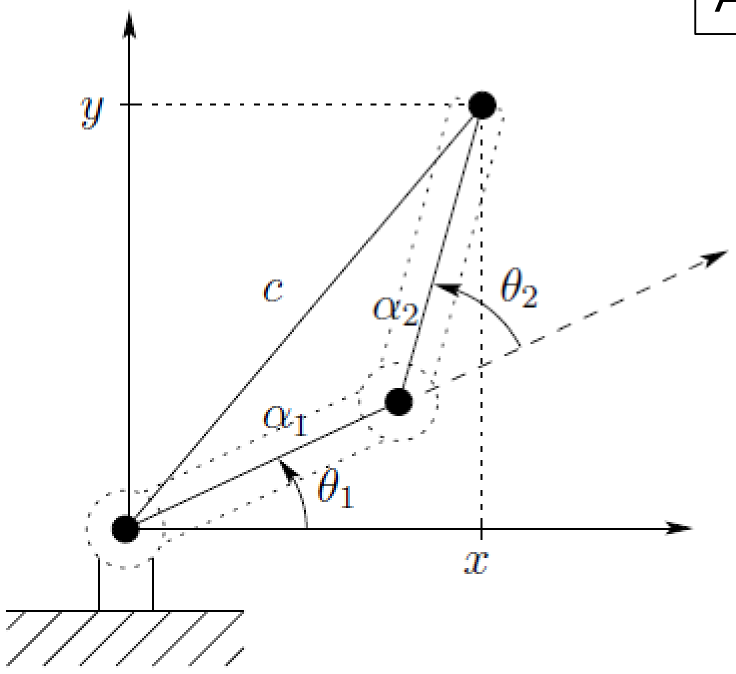
$$\vartheta_1 = \text{Atan2}(s_1, c_1). \qquad \vartheta_3 = \phi - \vartheta_1 - \vartheta_2.$$

Law of cosine explained

Articulated Configuration



$$a^2 = b^2 + c^2 - 2bc\cos(\theta)$$



$$c^2 = x^2 + y^2 = a_1^2 + a_2^2 - 2a_1a_2\cos(\pi - \theta_2)$$

$$c^2 = x^2 + y^2 = a_1^2 + a_2^2 + 2a_1a_2\cos(\theta_2)$$

$$\cos\theta_2 = \frac{x^2 + y^2 - \alpha_1^2 - \alpha_2^2}{2\alpha_1\alpha_2} = D.$$

Solution of Three-link Planar Arm

An alternative *geometric solution* technique is presented below.

The application of the cosine theorem to the triangle formed by links a_1, a_2 and the segment connecting points *W* and *O* gives

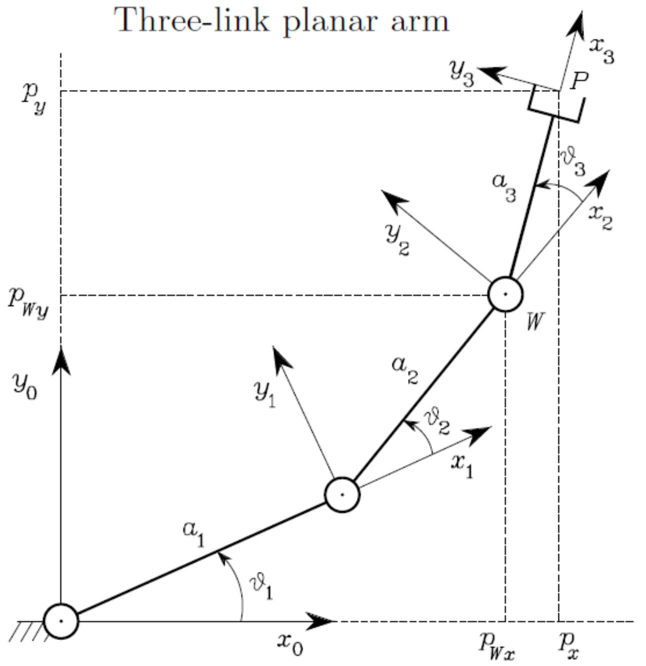
$$p_{Wx}^2 + p_{Wy}^2 = a_1^2 + a_2^2 - 2a_1a_2\cos(\pi - \vartheta_2)$$

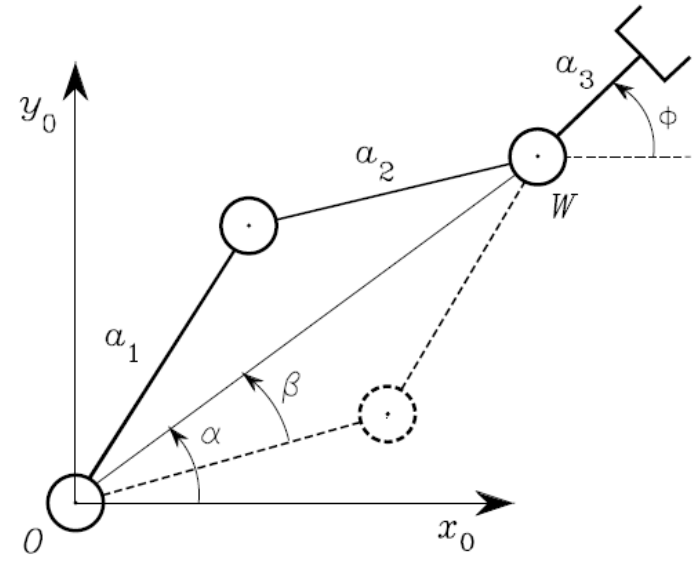
$\cos(\pi - \vartheta_2) = -\cos\vartheta_2$

$$c_2 = \frac{p_{Wx}^2 + p_{Wy}^2 - a_1^2 - a_2^2}{2a_1a_2}.$$

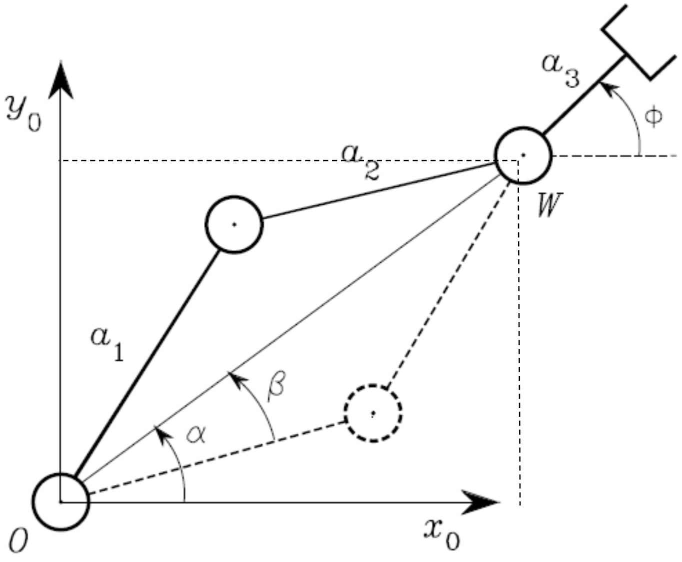
$$\vartheta_2 = \pm\cos^{-1}(c_2)$$

$$\begin{aligned} \text{elbow-up } & \vartheta_2 \in (-\pi, 0) \\ \text{elbow-down } & \vartheta_2 \in (0, \pi). \end{aligned}$$

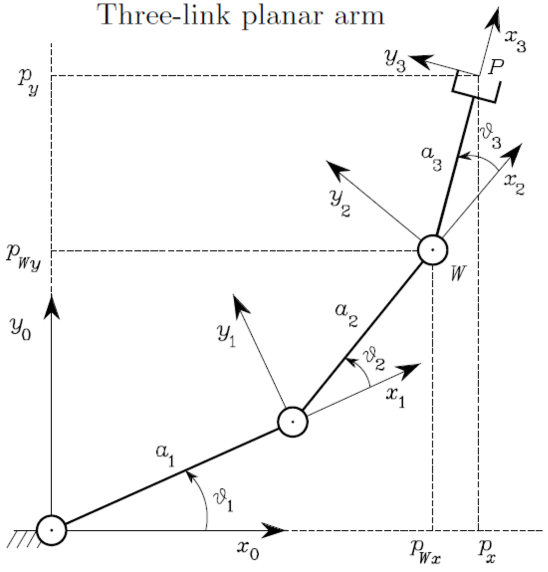




Solution of Three-link Planar Arm



Three-link planar arm



To find ϑ_1 consider the angles α and β

$$\alpha = \text{Atan2}(p_{Wy}, p_{Wx}).$$

To compute β, applying again the cosine theorem yields

$$c_\beta \sqrt{p_{Wx}^2 + p_{Wy}^2} = a_1 + a_2 c_2$$

and resorting to the expression of c_2 given above leads to

$$\beta = \cos^{-1}\left(\frac{p_{Wx}^2 + p_{Wy}^2 + a_1^2 - a_2^2}{2a_1\sqrt{p_{Wx}^2 + p_{Wy}^2}}\right)$$

$$\vartheta_1 = \alpha \pm \beta$$

$$\vartheta_3 = \phi - \vartheta_1 - \vartheta_2.$$

Solution of Spherical Wrist

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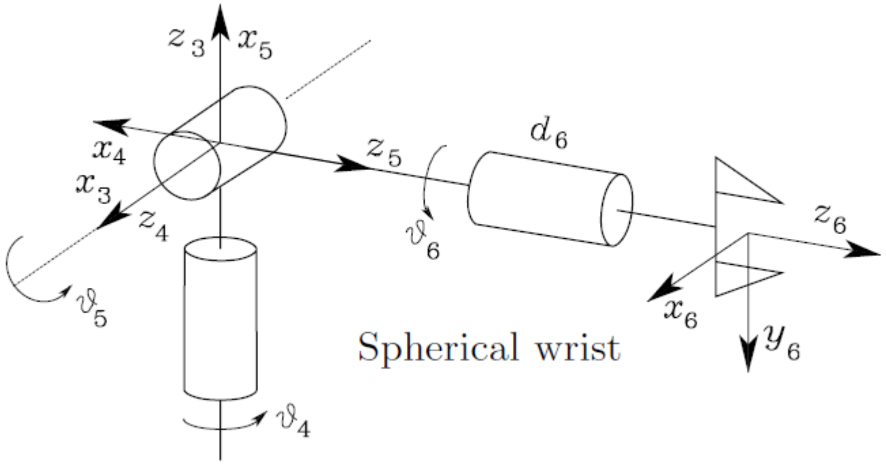


$$\boldsymbol{T}_6^3(\boldsymbol{q}) = \boldsymbol{A}_4^3\boldsymbol{A}_5^4\boldsymbol{A}_6^5 = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 & c_4s_5d_6 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 & s_4s_5d_6 \\ -s_5c_6 & s_5s_6 & c_5 & c_5d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{R}_6^3 = \begin{bmatrix} n_x^3 & s_x^3 & a_x^3 \\ n_y^3 & s_y^3 & a_y^3 \\ n_z^3 & s_z^3 & a_z^3 \end{bmatrix}$$

for $\vartheta_5 \in (0, \pi)$, and

$$\vartheta_4 = \text{Atan2}(a_y^3, a_x^3)$$
$$\vartheta_5 = \text{Atan2}\left(\sqrt{(a_x^3)^2 + (a_y^3)^2}, a_z^3\right)$$
$$\vartheta_6 = \text{Atan2}(s_z^3, -n_z^3)$$

for $\vartheta_5 \in (-\pi, 0)$

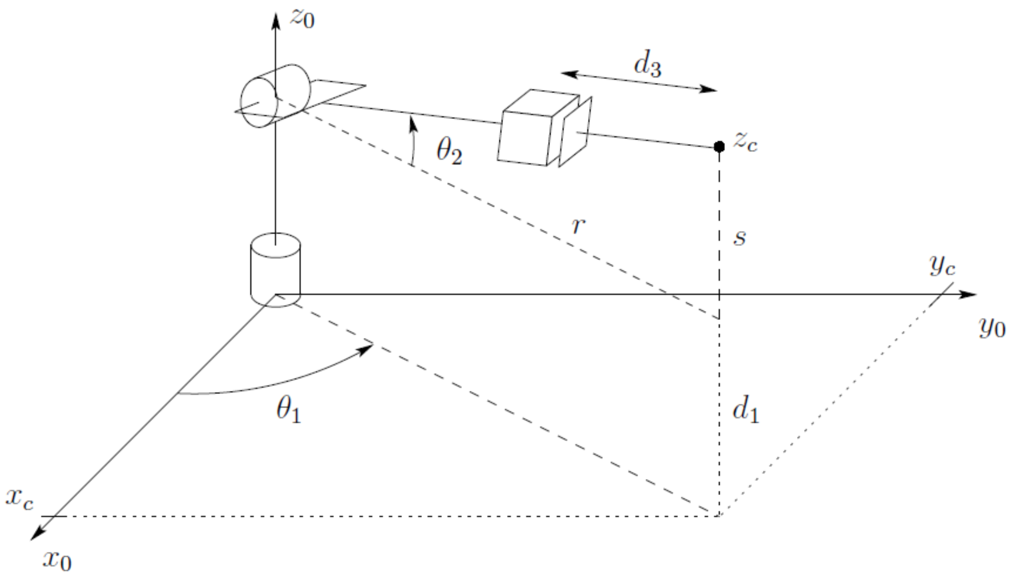
$$\vartheta_4 = \text{Atan2}(-a_y^3, -a_x^3)$$
$$\vartheta_5 = \text{Atan2}\left(-\sqrt{(a_x^3)^2 + (a_y^3)^2}, a_z^3\right)$$
$$\vartheta_6 = \text{Atan2}(-s_z^3, n_z^3)$$

DH parameters for spherical wrist.

Link	a_i	α_i	d_i	θ_i
4	0	–90	0	θ_4^*
5	0	90	0	θ_5^*
6	0	0	d_6	θ_6^*

Solution of Spherical manipulator

Spherical Configuration



As in the case of the elbow manipulator the first joint variable is the base rotation and a solution is given as

$$\theta_1 = A\tan(x_c, y_c)$$

provided x_c and y_c are not both zero.

The angle θ_2 is given from $$\theta_2 = A\tan(r, s) + \frac{\pi}{2}$$

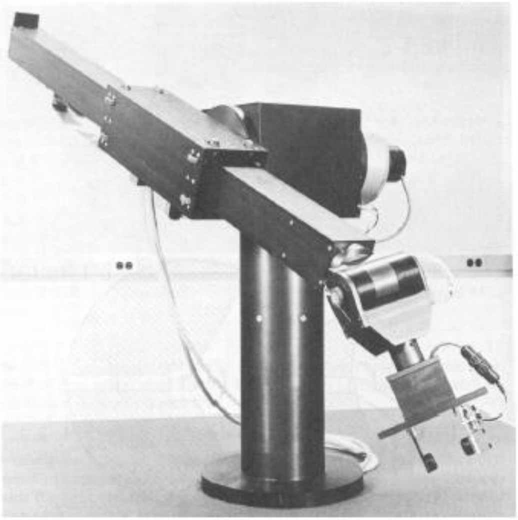
Where: $$r^2 = x_c^2 + y_c^2,\ s = z_c - d_1$$

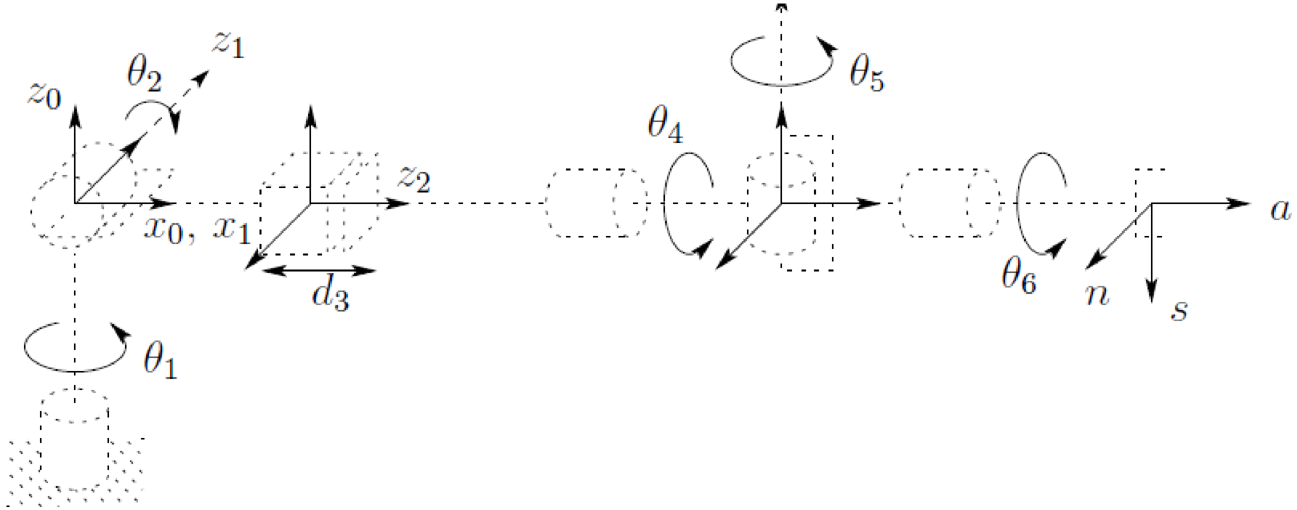
As in the case of the elbow manipulator a second solution for θ_1 is given by

$$\theta_1 = \pi + A\tan(x_c, y_c);$$

The linear distance d_3 is found as $$d_3 = \sqrt{r^2 + s^2} = \sqrt{x_c^2 + y_c^2 + (z_c - d_1)^2}.$$

Example





Recall the Stanford manipulator: Suppose that the desired position and orientation of the final frame are given by

$$H = \begin{bmatrix} r_{11} & r_{12} & r_{13} & o_x \\ r_{21} & r_{22} & r_{23} & o_y \\ r_{31} & r_{32} & r_{33} & o_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} R & o \\ 0 & 1 \end{bmatrix} \in SE(3)$$

To find the corresponding joint variables θ_1, θ_2, d_3, θ_4, θ_5, and θ_6 we must solve the following simultaneous set of nonlinear trigonometric equations

Example

$$
\begin{aligned}
c_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] - s_1(s_4c_5c_6 + c_4s_6) &= r_{11} \\
s_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] + c_1(s_4c_5c_6 + c_4s_6) &= r_{21} \\
-s_2(c_4c_5c_6 - s_4s_6) - c_2s_5s_6 &= r_{31} \\
c_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] - s_1(-s_4c_5s_6 + c_4c_6) &= r_{12} \\
s_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] + c_1(-s_4c_5s_6 + c_4c_6) &= r_{22} \\
s_2(c_4c_5s_6 + s_4c_6) + c_2s_5s_6 &= r_{32} \\
c_1(c_2c_4s_5 + s_2c_5) - s_1s_4s_5 &= r_{13} \\
s_1(c_2c_4s_5 + s_2c_5) + c_1s_4s_5 &= r_{23} \\
-s_2c_4s_5 + c_2c_5 &= r_{33} \\
c_1s_2d_3 - s_1d_2 + d_6(c_1c_2c_4s_5 + c_1c_5s_2 - s_1s_4s_5) &= o_x \\
s_1s_2d_3 + c_1d_2 + d_6(c_1s_4s_5 + c_2c_4s_1s_5 + c_5s_1s_2) &= o_y \\
c_2d_3 + d_6(c_2c_5 - c_4s_2s_5) &= o_z.
\end{aligned}
$$

Kinematic Decoupling

Although the general problem of inverse kinematics is quite difficult, it turns out that for manipulators having six joints, with the last three joints intersecting at a point (such as the Stanford Manipulator above), it is possible to decouple the inverse kinematics problem into two simpler problems, known respectively, as **inverse position kinematics**, and **inverse orientation kinematics**.

$$H = \begin{bmatrix} R & o \\ 0 & 1 \end{bmatrix} \in SE(3) \qquad T_n^0(q_1, \ldots, q_n) = H$$

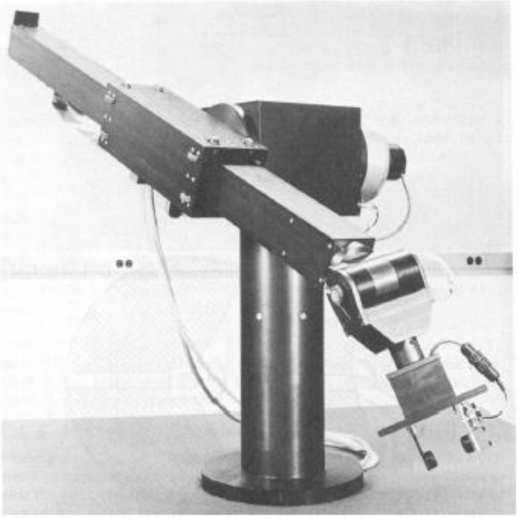
We express as two sets of equations representing the rotational and positional equations

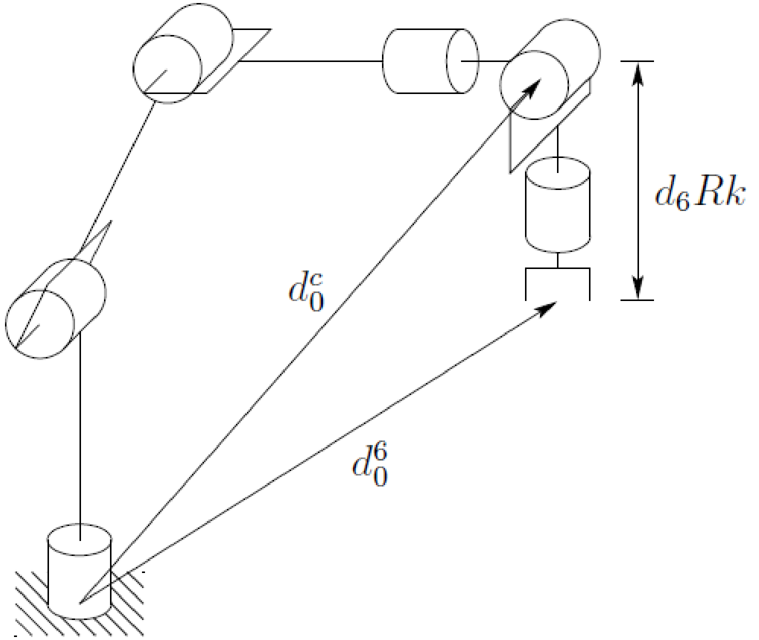
$$\begin{aligned} R_6^0(q_1, \ldots, q_6) &= R \\ o_6^0(q_1, \ldots, q_6) &= o \end{aligned}$$

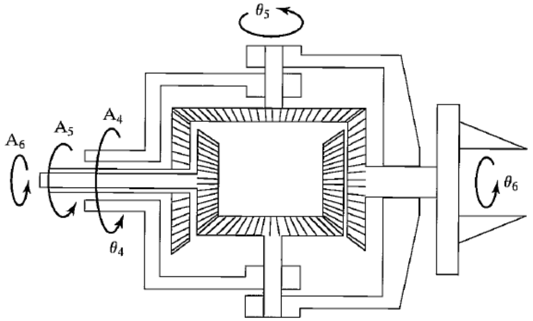
where o and R are the desired position and orientation of the tool frame

Kinematic Decoupling (example 1)

For concreteness let us suppose that there are exactly six degrees-of-freedom and that the last three joint axes intersect at a point o_c.

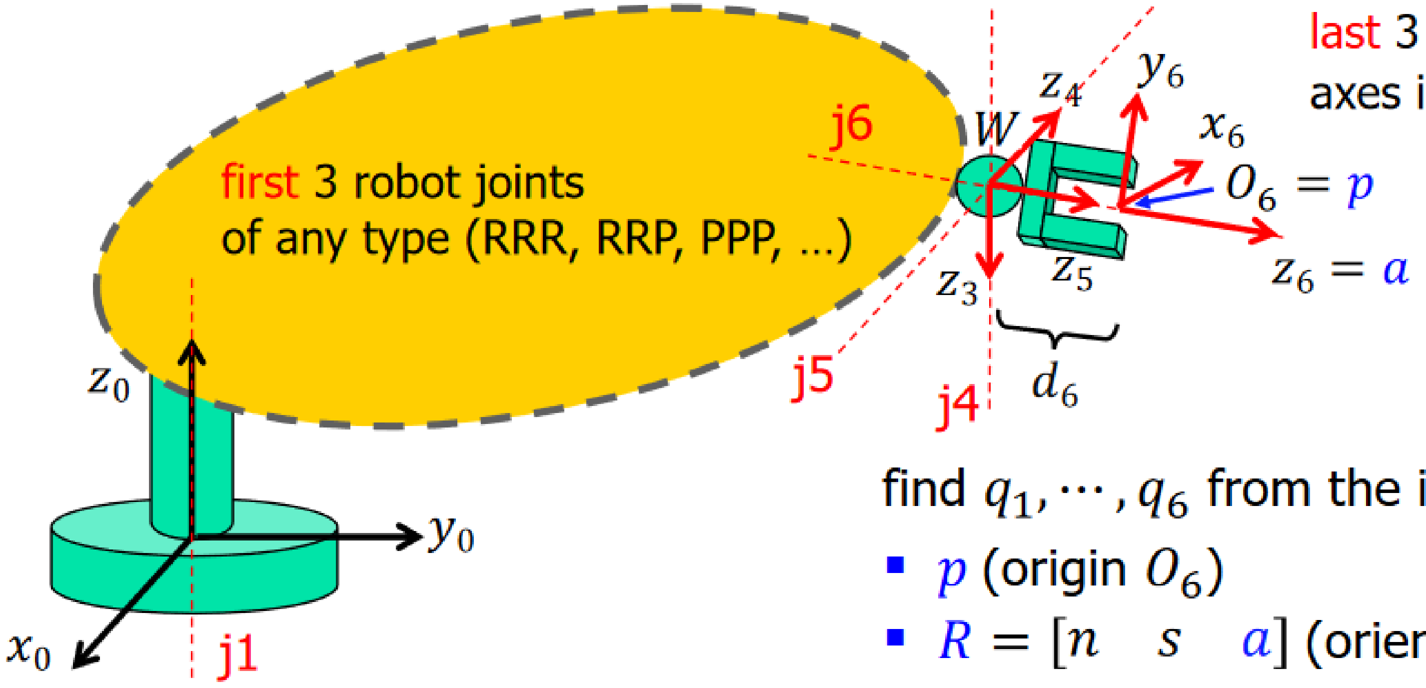






The important point of this assumption for the inverse kinematics is that motion of the final three links about these axes will not change the position of o_c, and thus, the position of the wrist center is thus a function of only the first three joint variables.

Inverse kinematics
for robots with spherical wrist



last 3 joints RRR, with axes intersecting in W

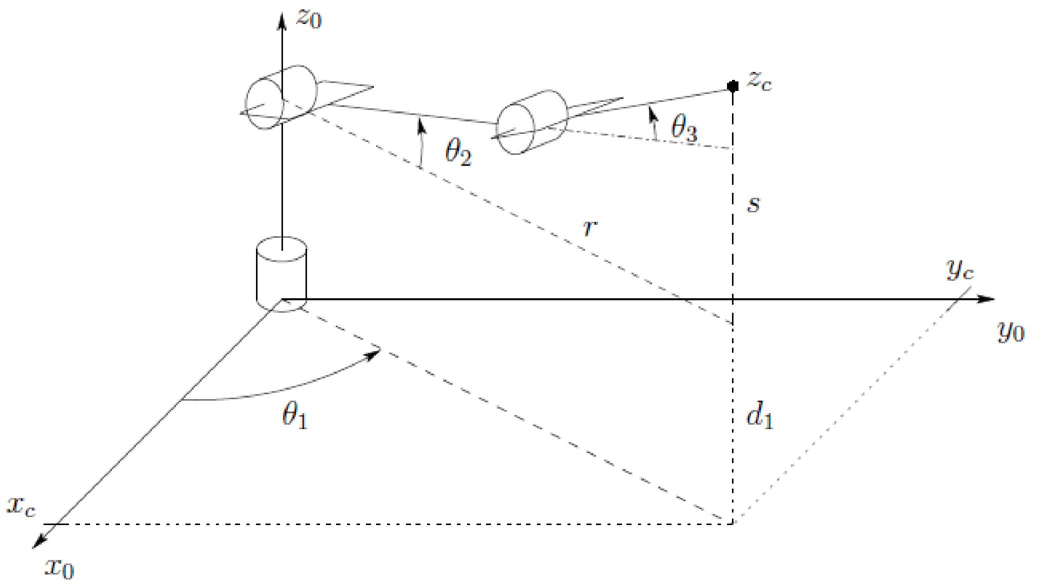
find $q_1, \cdots, q_6$ from the input data

- p (origin O_6)
- $R = [n \quad s \quad a]$ (orientation of RF_6)

1. $W = p - d_6 a \Rightarrow q_1, q_2, q_3$ (inverse "position" kinematics for main axes)
2. $R = {}^0R_3(q_1,q_2,q_3)\ {}^3R_6(q_4,q_5,q_6) \Rightarrow {}^3R_6(q_4,q_5,q_6) = {}^0R_3^T R \Rightarrow q_4, q_5, q_6$

   - R ↑ given
   - ${}^0R_3(q_1,q_2,q_3)$ ↑ known, after step 1
   - ${}^3R_6(q_4,q_5,q_6)$: Euler ZYZ or ZXZ rotation matrix with q_4,q_5,q_6 $(\theta_4,\theta_5,\theta_6)$ → two regular solutions
   - (inverse "orientation" kinematics for the wrist)

We decouple the two manipulators



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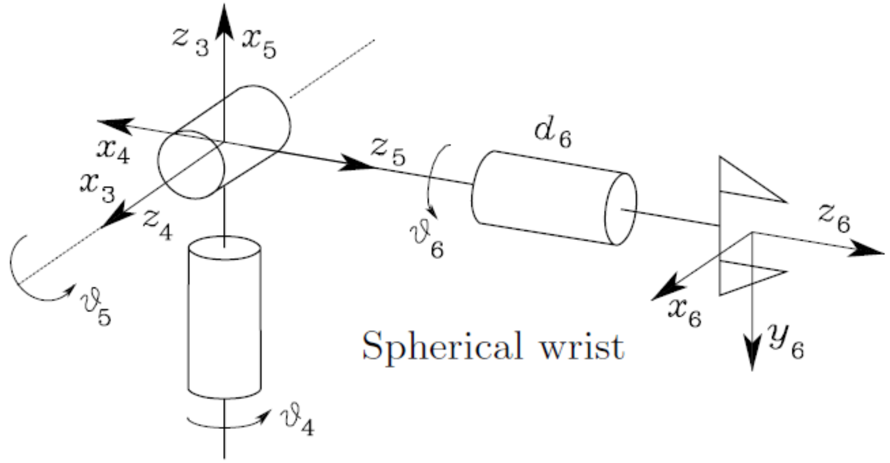

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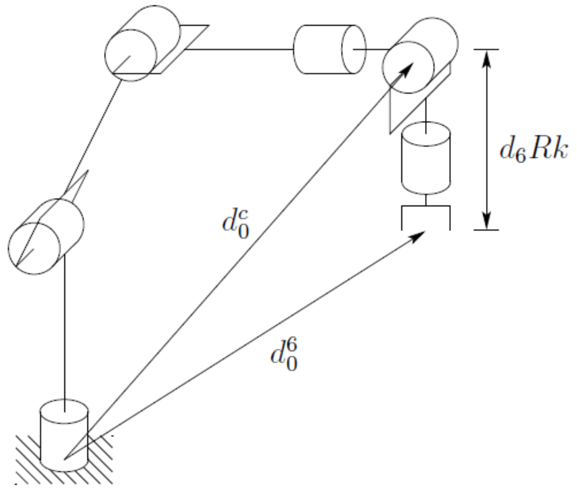
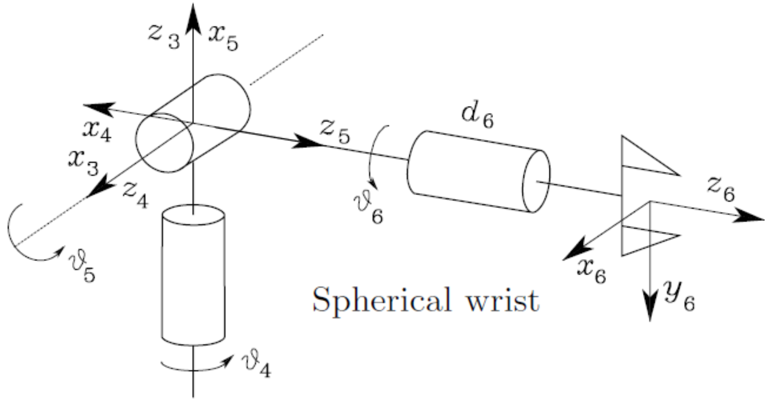


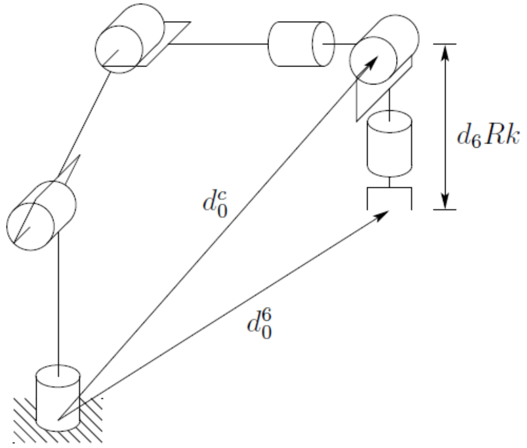
Kinematic Decoupling (example 1)

The origin of the tool frame (whose desired coordinates are given by o) is simply obtained by a translation of distance d_6 along z_5 from o_c .

DH parameters for spherical wrist.

Link	a_i	α_i	d_i	θ_i
4	0	-90	0	θ_4^*
5	0	90	0	θ_5^*
6	0	0	d_6	θ_6^*





In our case, z_5 and z_6 are the same axis, and the third column of R expresses the direction of z_6 with respect to the base frame. Therefore, we have

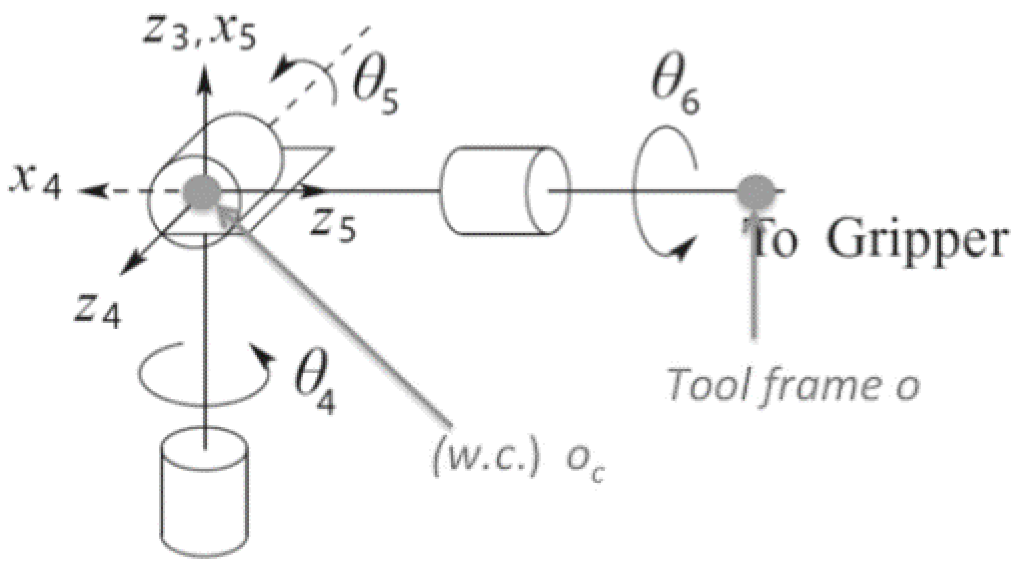
$$o = o_c^0 + d_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

(example 1)

Inverse Kinematics Trick: kinematic decoupling

Possible if 6 joints and last 3 joint axis intersect at a point

1. Find position of writs axes (*w.c.*) o_c
2. Find orientation of the wrist



DH parameters for spherical wrist.

Link	a_i	α_i	d_i	θ_i
4	0	−90	0	θ_4^*
5	0	90	0	θ_5^*
6	0	0	d_6	θ_6^*

It is only possible when a set of axis intersect in one point (axes 4 5 6) se also the next slide.

(example 1)

Solve with kinematic decoupling (position)

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$R_y = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem: given (R, o), solve for $q_1, \ldots, q_6$

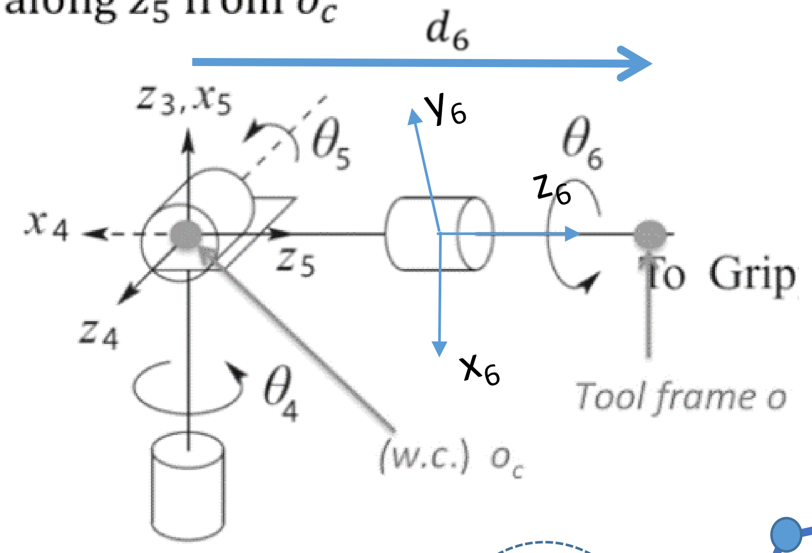
$$R_6^0(q_1, \ldots, q_6) = R$$
$$o_6^0(q_1, \ldots, q_6) = o$$

z_3, z_4, z_5 intersect at o_c, o_c is a function of q_1, q_2, q_3

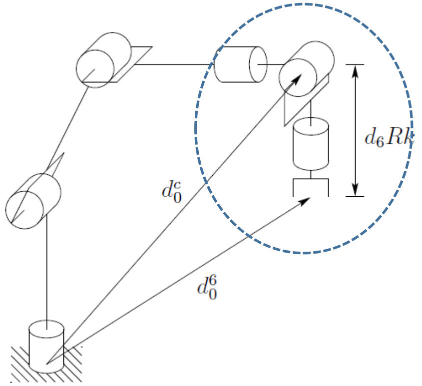
Tool frame origin o is translation d_6 along z_5 from o_c

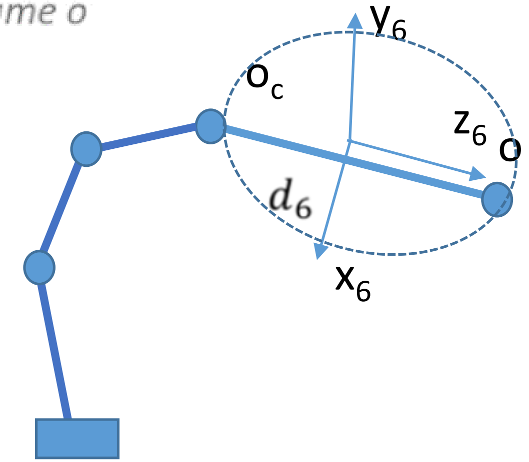
$$o = o_c^0 + d_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

1. **solve for** o_c^0



O is known from DH but we don't know O_c





(example 1)

Answer: kinematic decoupling (position)

Problem: given (R, o), solve for $q_1, \dots, q_6$

$R_6^0(q_1, \dots, q_6) = R, \quad o_6^0(q_1, \dots, q_6) = o$

z_3, z_4, z_5 intersect at o_c, o_c is a function of q_1, q_2, q_3

Tool frame origin o is translation d_6 along z_5 from o_c

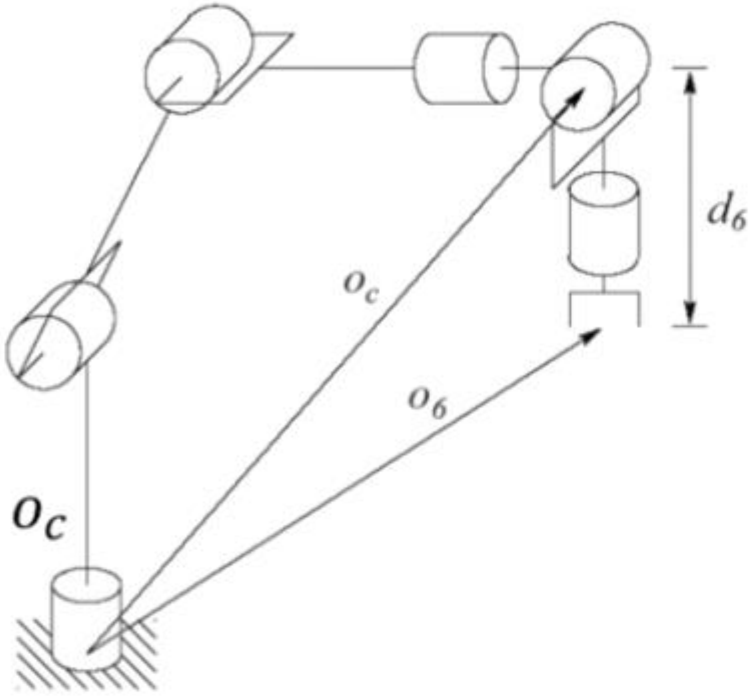
$$o = o_c^0 + d_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

1. **solve for** o_c^0

$$o_c^0 = o - d_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

2. $o = [o_x, o_y, o_z]^T$, $o_c = [x_c, y_c, z_c]^T$, **solve** o_c

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} o_x - d_6 r_{13} \\ o_y - d_6 r_{23} \\ o_z - d_6 r_{33} \end{bmatrix}$$



$$H = \begin{bmatrix} r_{11} & r_{12} & r_{13} & o_x \\ r_{21} & r_{22} & r_{23} & o_y \\ r_{31} & r_{32} & r_{33} & o_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Kinematic Decoupling

(example 1)

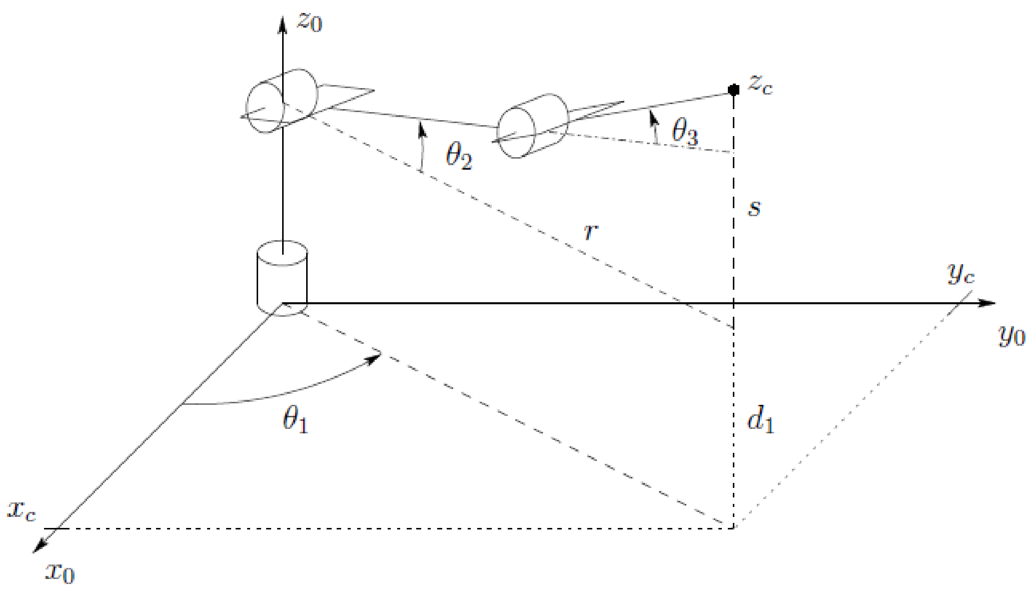
Thus in order to have the end-effector of the robot at the point with coordinates given by o and with the orientation of the end-effector given by $R = (r_{ij})$, it is necessary and sufficient that the wrist center o_c have coordinates given by

$$o_c^0 = o - d_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

and that the orientation of the frame $o_6x_6y_6z_6$ with respect to the base be given by R. If the components of the end-effector position o are denoted o_x, o_y, o_z and the components of the wrist center o_c^0 are denoted x_c, y_c, z_c then

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} o_x - d_6 r_{13} \\ o_y - d_6 r_{23} \\ o_z - d_6 r_{33} \end{bmatrix}.$$

How to evaluate the three angles?



$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} o_x - d_6 r_{13} \\ o_y - d_6 r_{23} \\ o_z - d_6 r_{33} \end{bmatrix}$$

See the example for Inverse Kinematics for the Articulate Elbow (Tutorials)

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Answer kinematic decoupling (orientation)

$$H = \begin{bmatrix} r_{11} & r_{12} & r_{13} & o_x \\ r_{21} & r_{22} & r_{23} & o_y \\ r_{31} & r_{32} & r_{33} & o_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

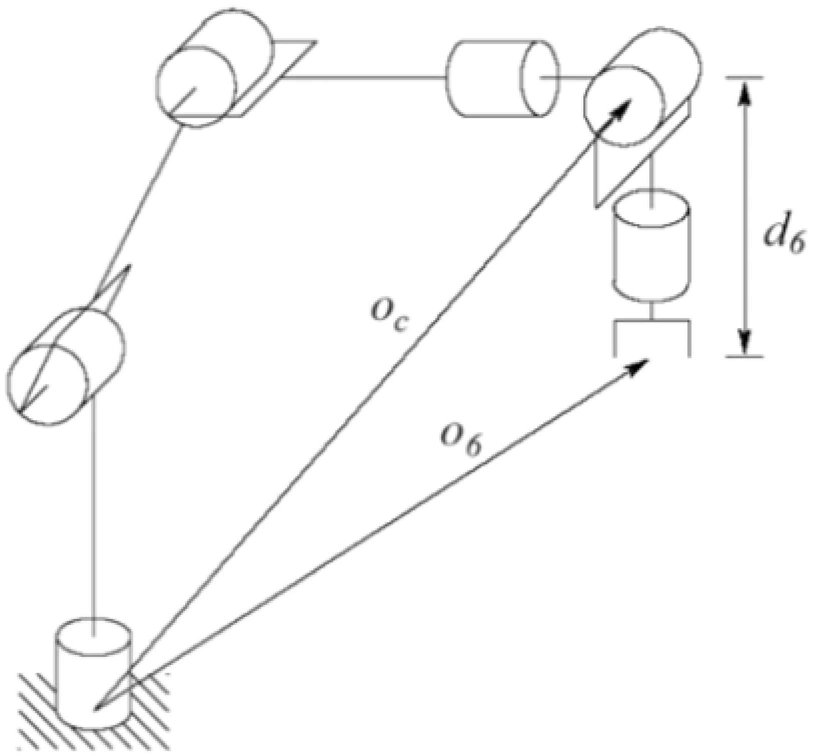
To get orientation of end effector relative to $o_3x_3y_3z_3$,

$$R = R_3^0 R_6^3$$

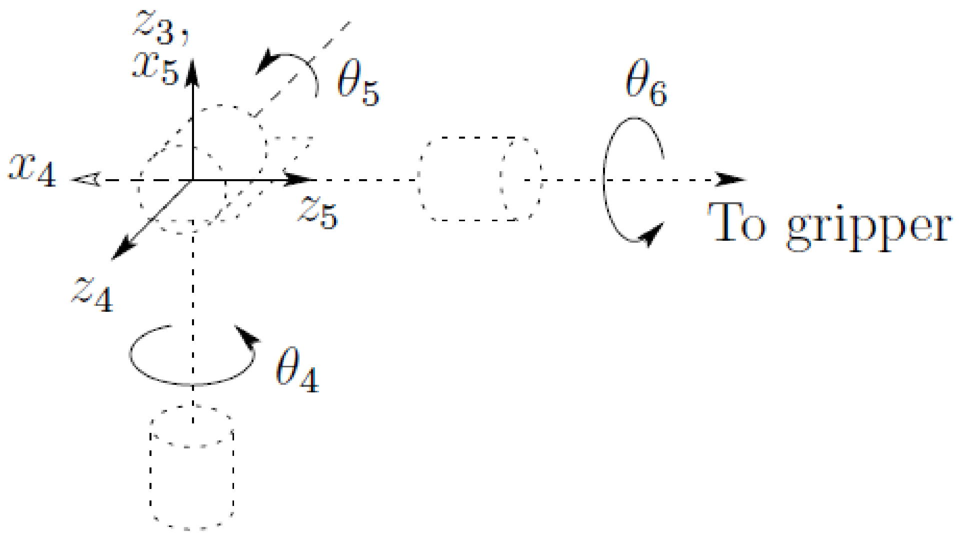
Solve

$$R_6^3 = (R_3^0)^{-1} R = (R_3^0)^{-1} R_3^0 R_6^3$$
$$= (R_3^0)^T R$$

Final 3 joint angles solve Euler angles for R_6^3



Remember: Spherical Wrist



Link	a_i	α_i	d_i	θ_i
4	0	−90	0	θ_4^*
5	0	90	0	θ_5^*
6	0	0	d_6	θ_6^*

* variable

Coordinates of the end-effector respect to the base (in this case is link 3 the base which is not visible)

$$T_6^3 = A_4A_5A_6 = \begin{bmatrix} R_6^3 & o_6^3 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 & c_4s_5d_6 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 & s_4s_5d_6 \\ -s_5c_6 & s_5s_6 & c_5 & c_5d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation of the end-effector around the frame x4 y4 z4

kinematic decoupling (orientation)

From Euler Angle (lecture 3)

$$R_{ZYZ}=\begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

From Euler Angle spherical wrist (lecture 3)

$$T_6^3 = A_4 A_5 A_6 = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -s_4c_6 - c_4c_5s_6 & c_4s_5 & d_6^*c_4s_5 \\ c_5c_6s_4 + c_4s_6 & c_4c_6 - c_5s_4s_6 & s_4s_5 & d_6^*s_4s_5 \\ -c_6s_5 & s_5s_6 & c_5 & d_6^*c_5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Let $\phi = \theta_4^*, \theta = \theta_5^*,$ and $\psi = \theta_6^*$

From Lecture 2 Inverse problem

$$\boldsymbol{R}(\phi) = \boldsymbol{R}_z(\varphi)\boldsymbol{R}_{y'}(\vartheta)\boldsymbol{R}_{z''}(\psi) = \begin{bmatrix} c_\varphi c_\vartheta c_\psi - s_\varphi s_\psi & -c_\varphi c_\vartheta s_\psi - s_\varphi c_\psi & c_\varphi s_\vartheta \\ s_\varphi c_\vartheta c_\psi + c_\varphi s_\psi & -s_\varphi c_\vartheta s_\psi + c_\varphi c_\psi & s_\varphi s_\vartheta \\ -s_\vartheta c_\psi & s_\vartheta s_\psi & c_\vartheta \end{bmatrix}$$

It is useful to solve the *inverse problem*, that is to determine the **set of Euler** angles corresponding to a given rotation matrix (known)

$$\boldsymbol{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

By considering the elements [1, 3] and [2, 3] $\varphi = \mathrm{Atan2}(r_{23}, r_{13})$

The function $\mathrm{Atan2}(y, x)$ computes the arctangent of the ratio y/x but utilizes the sign of each argument to determine which quadrant the resulting angle belongs to; this allows the correct determination of an angle in a range of 2π.

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From Lecture 2 Inverse problem

Then, squaring and summing the elements [1, 3] and [2, 3] and using the element [3, 3] yields

$$\vartheta = \mathrm{Atan2}\left(\sqrt{r_{13}^2 + r_{23}^2}, r_{33}\right)$$

The choice of the positive sign for the term $r^2_{13} + r^2_{23}$ limits the range of feasible values of ϑ to $(0, \pi)$.

On this assumption, considering the elements [3, 1] and [3, 2] gives

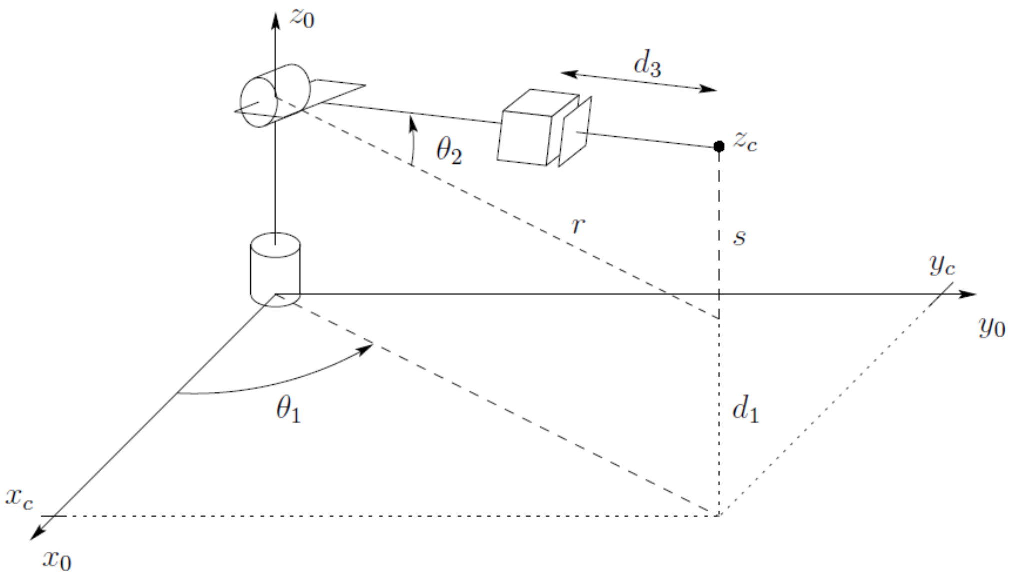
$$\psi = \mathrm{Atan2}(r_{32}, -r_{31})$$

Kinematic decoupling

- Allows to simplify complex problem by dividing it in two main structures of known kinematics (From DH usually)
- It works only in case of spherical wrists with the three axes which are coincident
- You have anyway to solve two inverse kinematic problems (manipulator and spherical wrist)
- In general we can use a geometric approach on few robotic structures but for the more complex ones we need numerical methods (next class)

Solution of Spherical Manipulator

Spherical Configuration



As in the case of the elbow manipulator the first joint variable is the base rotation and a solution is given as

$$\theta_1 \quad = \quad A\tan(x_c, y_c)$$

provided x_c and y_c are not both zero.

The angle θ_2 is given from $$\theta_2 \quad = \quad A\tan(r, s) + \frac{\pi}{2}$$

Where: $$r^2 = x_c^2 + y_c^2,\; s = z_c - d_1$$

As in the case of the elbow manipulator a second solution for θ_1 is given by

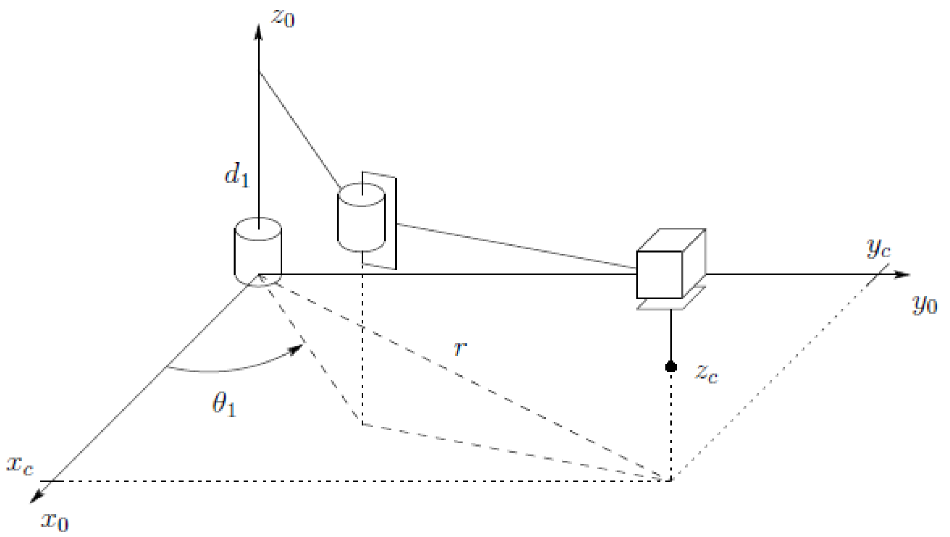
$$\theta_1 = \pi + A\tan(x_c, y_c);$$

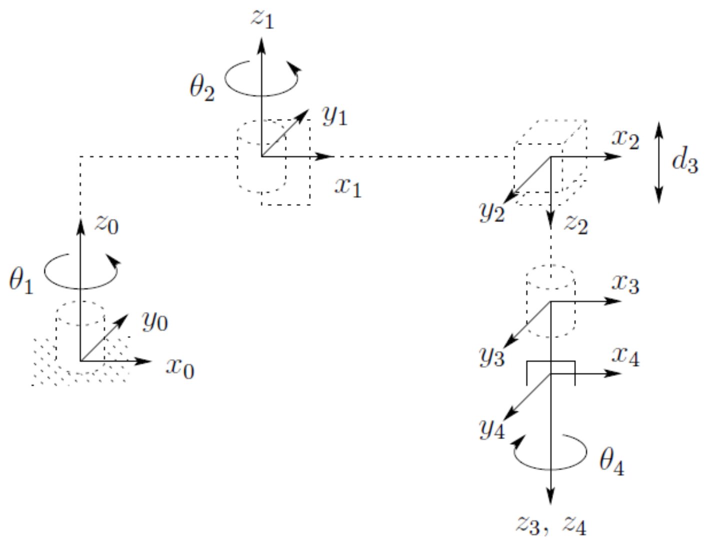
The linear distance d_3 is found as $$d_3 = \sqrt{r^2 + s^2} \quad = \quad \sqrt{x_c^2 + y_c^2 + (z_c - d_1)^2}.$$

Solution of SCARA

SCARA Manipulator

As another example, we consider the SCARA manipulator whose forward kinematics is defined by T_4^0 .





Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	θ_1
2	a_2	180	0	θ_2
3	0	0	d_3	0
4	0	0	d_4	θ_4

$$T_4^0 = A_1 \cdots A_4 = \begin{bmatrix} c_{12}c_4 + s_{12}s_4 & -c_{12}s_4 + s_{12}c_4 & 0 & a_1c_1 + a_2c_{12} \\ s_{12}c_4 - c_{12}s_4 & -s_{12}s_4 - c_{12}c_4 & 0 & a_1s_1 + a_2s_{12} \\ 0 & 0 & -1 & -d_3 - d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

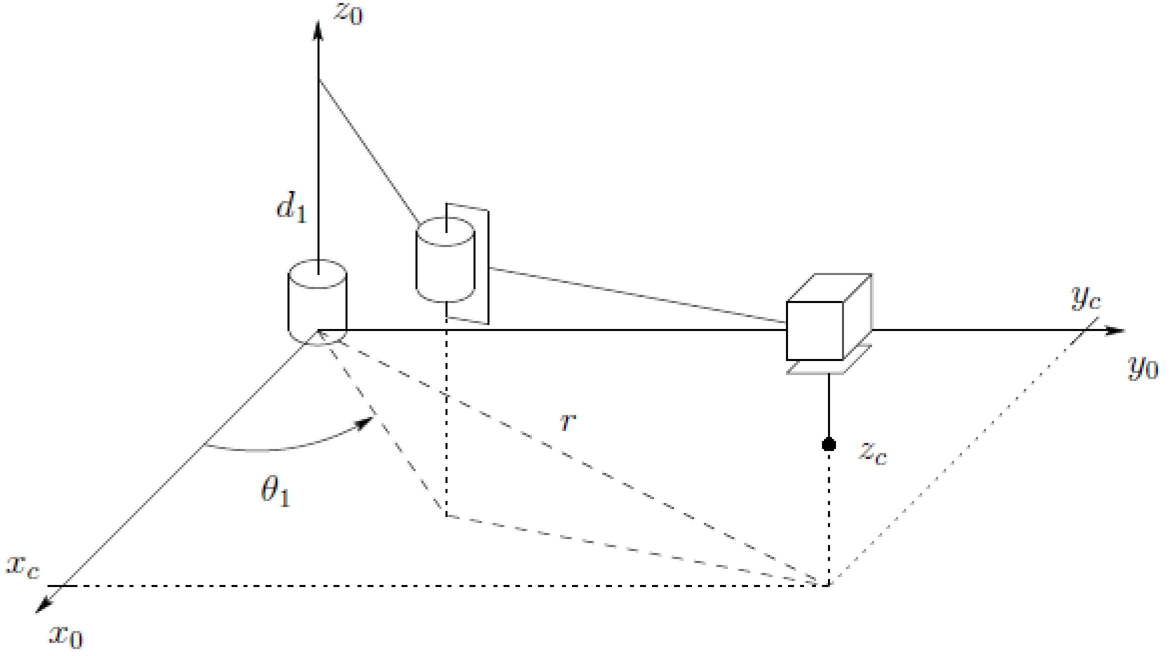
Solution of SCARA

SCARA Manipulator

The inverse kinematics is then given as the set of solutions of the equation

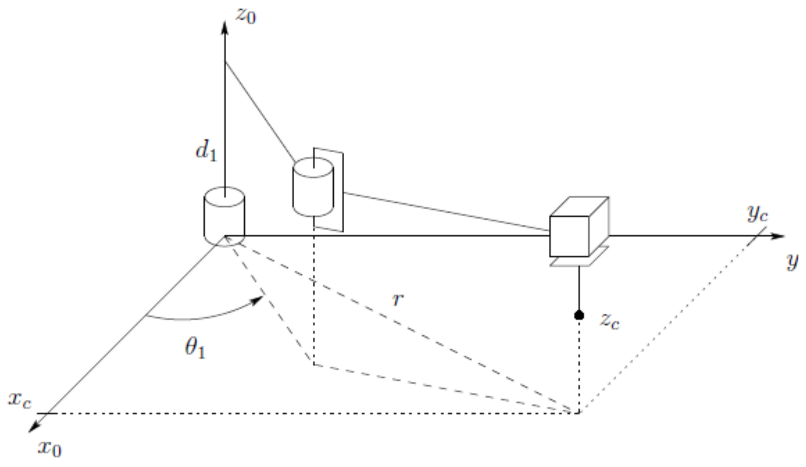
$$\begin{bmatrix} c_{12}c_4 + s_{12}s_4 & s_{12}c_4 - c_{12}s_4 & 0 & a_1c_1 + a_2c_{12} \\ s_{12}c_4 - c_{12}s_4 & -c_{12}c_4 - s_{12}s_4 & 0 & a_1s_1 + a_2s_{12} \\ 0 & 0 & -1 & -d_3 - d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} R & o \\ 0 & 1 \end{bmatrix}.$$

Projecting the manipulator configuration onto the $x_0 - y_0$ *plane immediately yields the situation of Figure*



Solution of SCARA

SCARA Manipulator



The transformation from the base 0 to the end effector 4 is a rotation matrix given by:

$$R = \begin{bmatrix} c_\alpha & s_\alpha & 0 \\ s_\alpha & -c_\alpha & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad \theta_1 + \theta_2 - \theta_4 = \alpha = A\tan(r_{11}, r_{12})$$

We see from this that $\theta_2 = A\tan\left(c_2, \pm\sqrt{1 - c_2}\right)$

where $c_2 = \dfrac{o_x^2 + o_y^2 - a_1^2 - a_2^2}{2a_1 a_2}$

$$\theta_1 = A\tan(o_x, o_y) - A\tan(a_1 + a_2 c_2, a_2 s_2).$$

We may then determine θ_4 *from*

$$\theta_4 = \theta_1 + \theta_2 - \alpha = \theta_1 + \theta_2 - A\tan(r_{11}, r_{12}).$$

Finally d_3 *is given as* $d_3 = o_z + d_4.$

The end!