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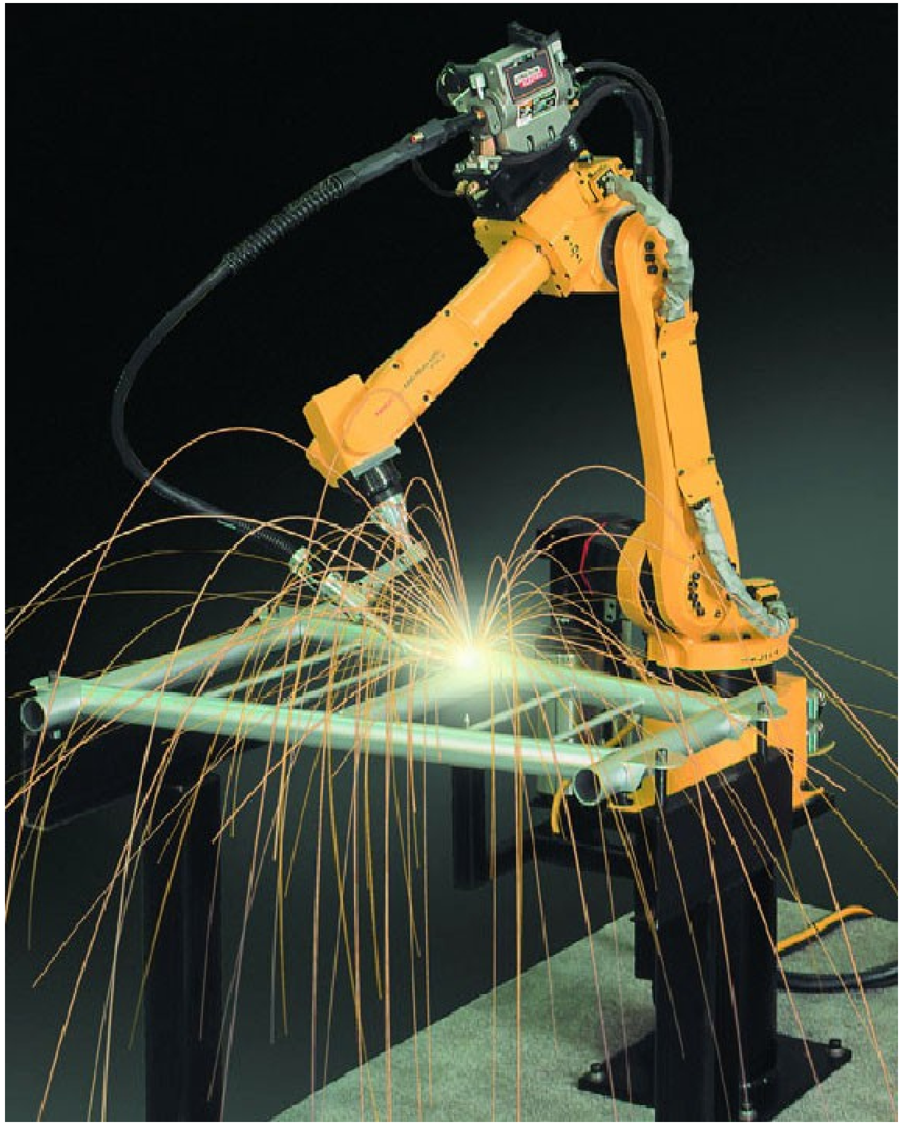
DIFFERENTIAL KINEMATICS



VELOCITY KINEMATICS – THE MANIPULATOR JACOBIAN

- In the previous classes we derived the forward and inverse position equations relating joint positions and end-effector positions and orientations.

- In this class we derive the velocity relationships, relating the linear and angular velocities of the end-effector (or any other point on the manipulator) to the joint velocities.

- In particular, we will derive the angular velocity of the end-effector frame (which gives the rate of rotation of the frame) and the linear velocity of the origin.

- The velocity relationships are then determined by the **Jacobian matrix**.

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Jacobian matrix

Configuration space (n=2)

Task space (m=3)

$$\boldsymbol{J}(\theta_1, \theta_2) = \begin{pmatrix} -d_1 \sin(\theta_1) - d_2 \sin(\theta_1 + \theta_2) & -d_2 \sin(\theta_1 + \theta_2) \\ d_1 \cos(\theta_1) + d_2 \cos(\theta_1 + \theta_2) & d_2 \cos(\theta_1 + \theta_2) \\ 1 & 1 \end{pmatrix}$$

Remarks:

- $\boldsymbol{J}$ depends on the joint angles (θ_1, θ_2);
- $\boldsymbol{J}$ has as many columns as the number of joint angles (here: 2), and as many rows as the number of parameters of the end-effector (here: 3).

The Jacobian matrix is useful in that it gives the relationship between joint angle velocity $\dot{\boldsymbol{q}}$ and the end-effector velocity $\dot{\boldsymbol{p}}$:

$$\dot{\boldsymbol{p}} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}.$$

THE MANIPULATOR JACOBIAN

This Jacobian or Jacobian matrix is one of the most important quantities in the analysis and control of robot motion.

It arises in virtually every aspect of robotic manipulation:

1. in the planning and execution of smooth trajectories,
2. in the determination of singular configurations,
3. in the execution of coordinated anthropomorphic motion,
4. in the derivation of the dynamic equations of motion,
5. and in the transformation of forces and torques from **the end-effector to the manipulator joints.**

Angular Velocity

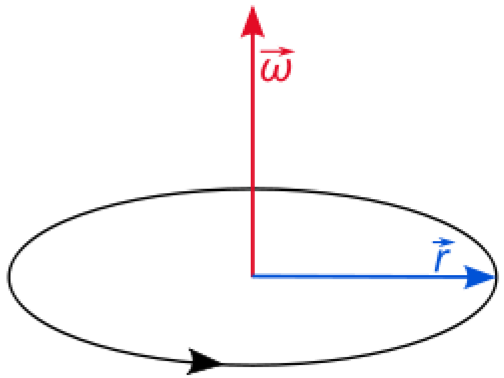
When a rigid body moves in a pure rotation about a fixed axis, every point of the body moves in a circle, then the angular velocity is given by

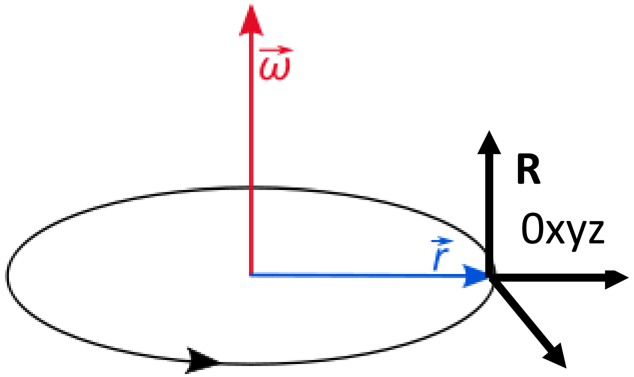
$$\omega = \dot{\theta}\boldsymbol{k}$$

k is a unit vector in the direction of the axis of rotation.

Given the angular velocity of the body, one learns in introductory dynamics courses that the linear velocity of any point on the body is given by the equation

$$\boldsymbol{v} = \omega \times \boldsymbol{r}$$



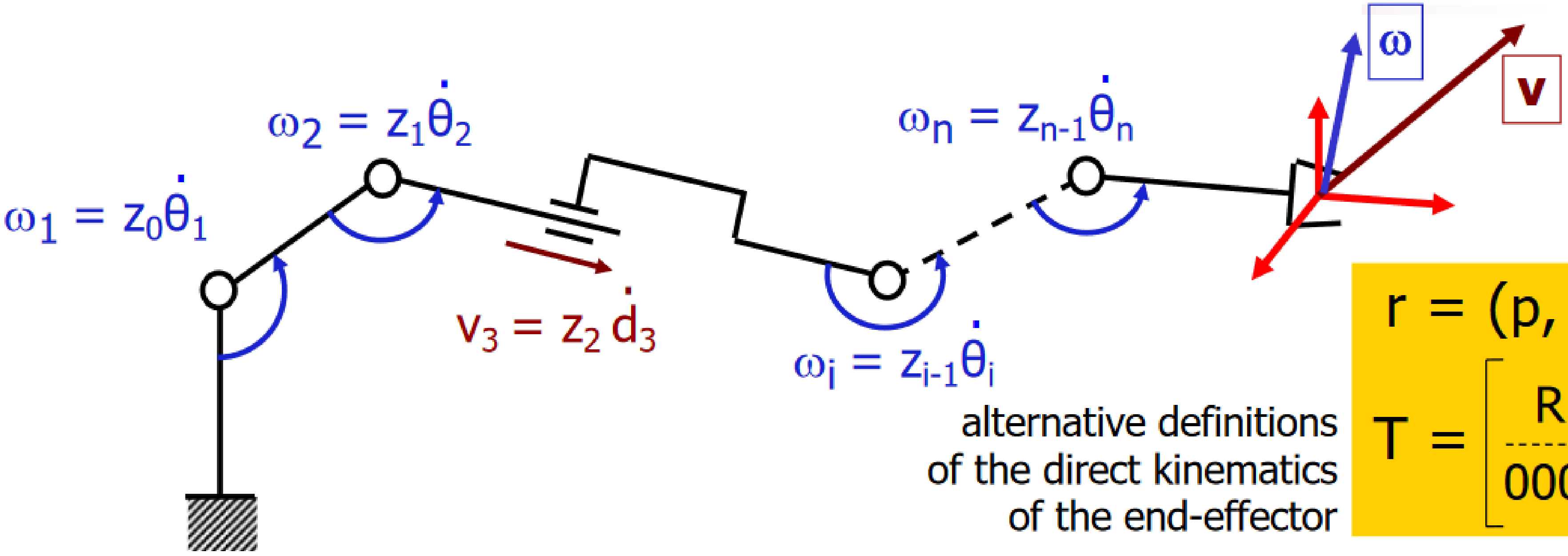


If we attach a body **R**: Since every point on the object experiences the same angular velocity and since each point of the body is in a fixed geometric relationship to the body-attached frame, the angular velocity is a property of the attached coordinate frame itself.

Differential kinematics

- "relations between motion (velocity) in joint space and motion (linear/angular velocity) in task space (e.g., Cartesian space)"
- instantaneous velocity mappings can be obtained through time derivation of the direct kinematics or in a geometric way, directly at the differential level
  - different treatments arise for rotational quantities
  - establish the link between angular velocity and
    - time derivative of a rotation matrix
    - time derivative of the angles in a minimal representation of orientation

Linear and angular velocity of the robot end-effector



$$r = (p, \phi)$$

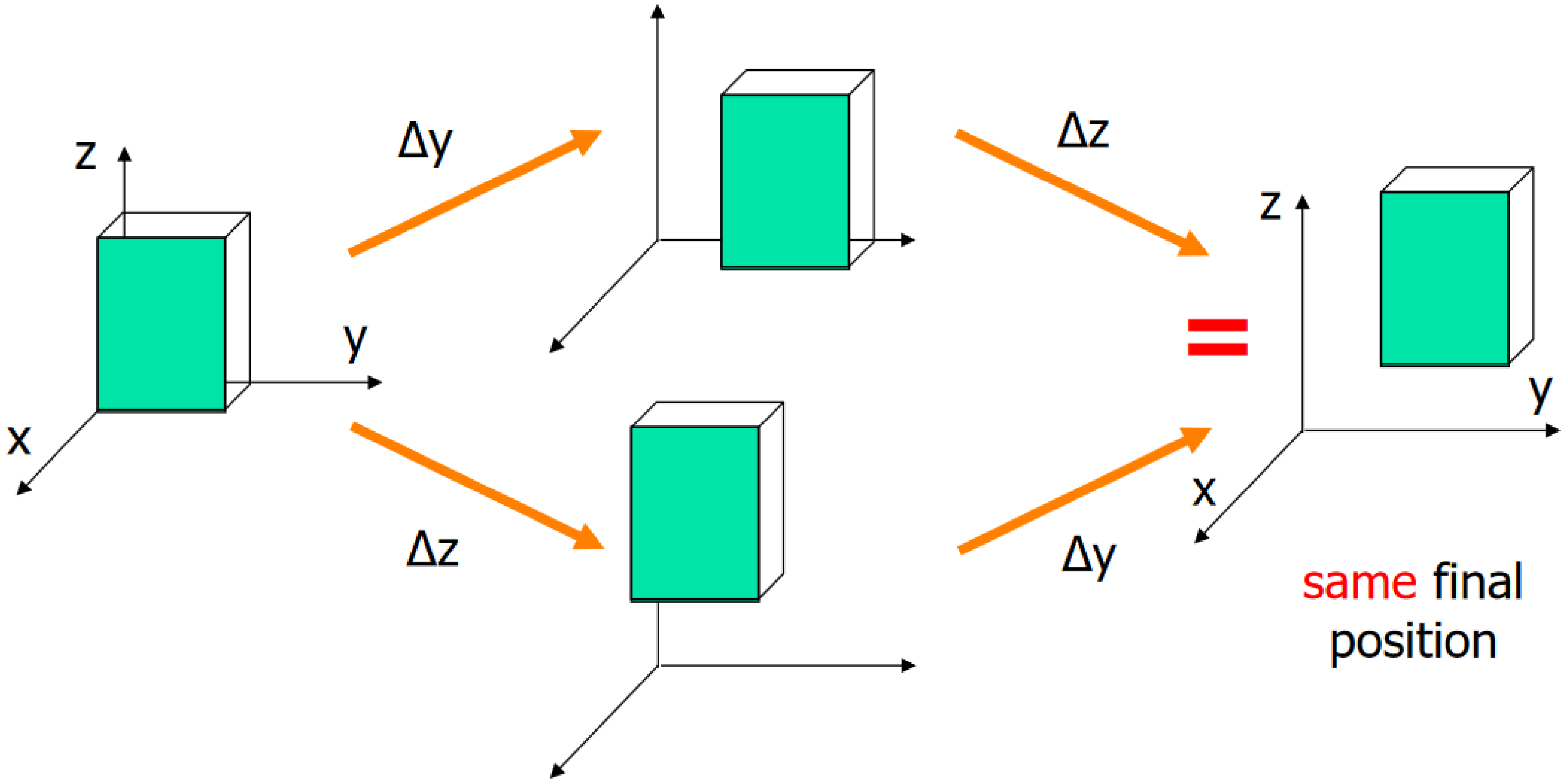
$$T = \left[\begin{array}{c|c} R & p \\ \hline 000 & 1 \end{array}\right]$$

- v and ω are "vectors", namely are elements of vector spaces
  - they can be obtained as the sum of single contributions (in any order)
  - these contributions will be those of the single the joint velocities
- on the other hand, ϕ (and $\dot{\phi}$) is not an element of a vector space
  - a minimal representation of a sequence of two rotations is not obtained summing the corresponding minimal representations (accordingly, for their time derivatives)

in general, $\omega \neq \dot{\phi}$

Finite and infinitesimal translations

- finite $\Delta x, \Delta y, \Delta z$ or infinitesimal dx, dy, dz translations (linear displacements) always commute



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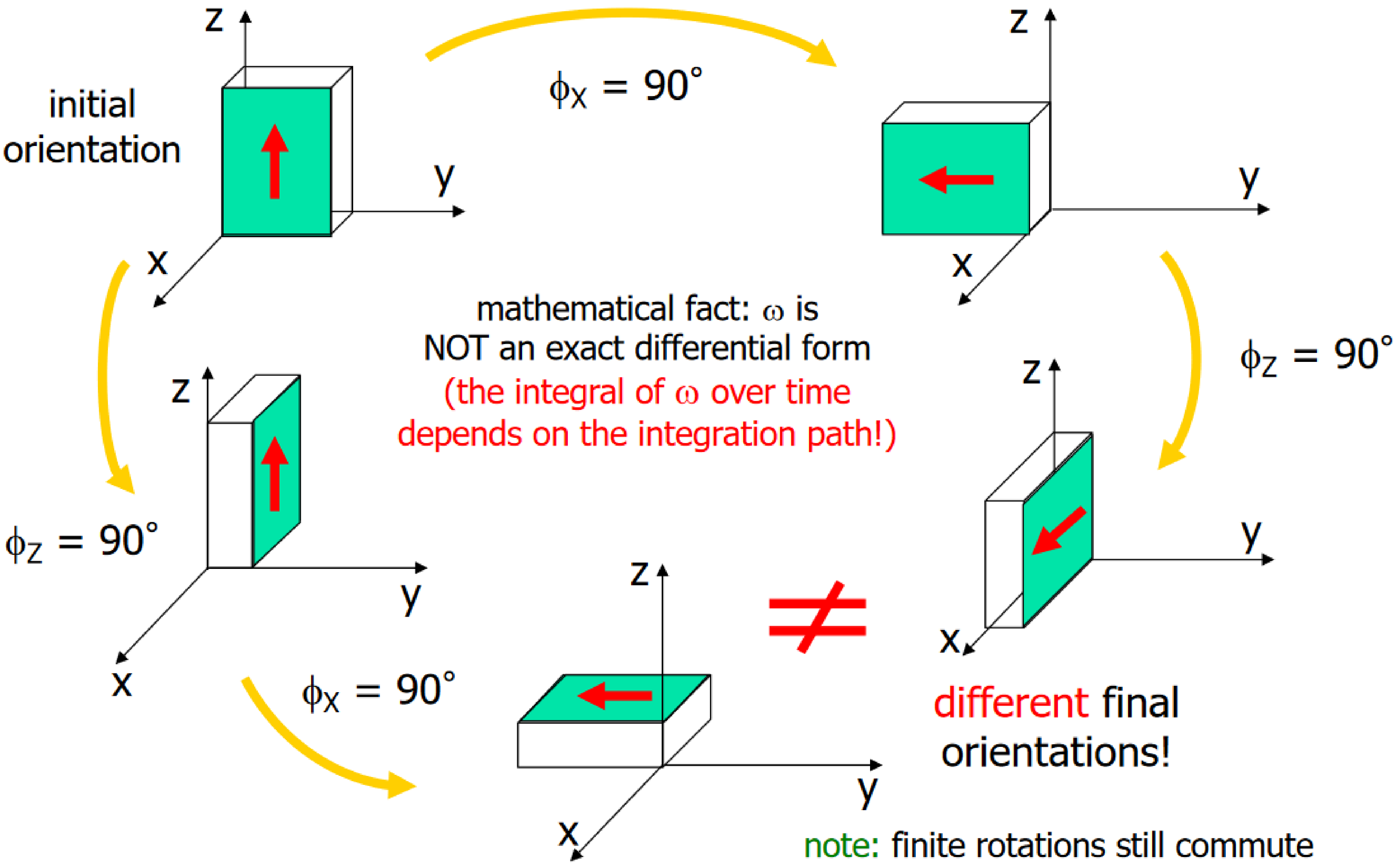
Finite rotations do not commute

example

Demonstration 1
on the black board
after the slide



mathematical fact: ω is
NOT an exact differential form
(the integral of ω over time
depends on the integration path!)

note: finite rotations still commute
when made around the same fixed axis

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Infinitesimal rotations commute!

- infinitesimal rotations $d\phi_X, d\phi_Y, d\phi_Z$ around x, y, z axes

Demonstration 2
on the black board

$$R_X(\phi_X) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_X & -\sin\phi_X \\ 0 & \sin\phi_X & \cos\phi_X \end{bmatrix} \Rightarrow R_X(d\phi_X) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -d\phi_X \\ 0 & d\phi_X & 1 \end{bmatrix}$$

$$R_Y(\phi_Y) = \begin{bmatrix} \cos\phi_Y & 0 & \sin\phi_Y \\ 0 & 1 & 0 \\ -\sin\phi_Y & 0 & \cos\phi_Y \end{bmatrix} \Rightarrow R_Y(d\phi_Y) = \begin{bmatrix} 1 & 0 & d\phi_Y \\ 0 & 1 & 0 \\ -d\phi_Y & 0 & 1 \end{bmatrix}$$

$$R_Z(\phi_Z) = \begin{bmatrix} \cos\phi_Z & -\sin\phi_Z & 0 \\ \sin\phi_Z & \cos\phi_Z & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow R_Z(d\phi_Z) = \begin{bmatrix} 1 & -d\phi_Z & 0 \\ d\phi_Z & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- $$R(d\phi) = R(d\phi_X, d\phi_Y, d\phi_Z) = \begin{bmatrix} 1 & -d\phi_Z & d\phi_Y \\ d\phi_Z & 1 & -d\phi_X \\ -d\phi_Y & d\phi_X & 1 \end{bmatrix} = I + S(d\phi)$$

↑ in any order

← neglecting second- and third-order (infinitesimal) terms

skew-symmetric matrix

In mathematics, particularly in linear algebra, a **skew-symmetric** (or **antisymmetric** or **antimetric**[1]) **matrix** is a square matrix whose transpose equals its negative, that is, it satisfies the condition[2]:p. 38

$$A \text{ skew-symmetric} \quad \iff \quad A^{\mathsf{T}} = -A$$

In terms of the entries of the matrix, if a_{ij} denotes the entry in the i-th row and j-th column, then the skew-symmetric condition is equivalent to

$$A \text{ skew-symmetric} \quad \iff \quad a_{ji} = -a_{ij}$$

Example

For example, the following matrix is skew-symmetric:

$$A = \begin{bmatrix} 0 & 2 & -1 \\ -2 & 0 & -4 \\ 1 & 4 & 0 \end{bmatrix}$$

because

$$-A = \begin{bmatrix} 0 & -2 & 1 \\ 2 & 0 & 4 \\ -1 & -4 & 0 \end{bmatrix} = A^{\mathsf{T}}$$

Time derivative of a rotation matrix

Demonstration 3

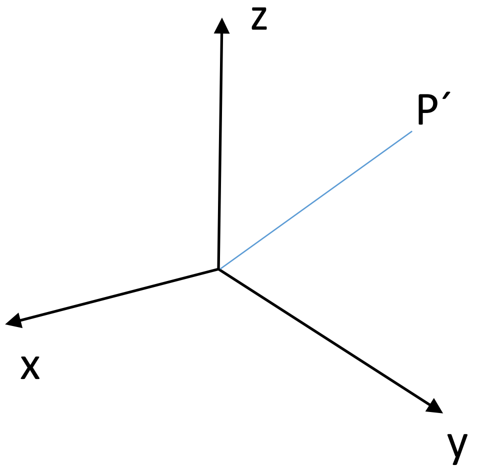
- let $R = R(t)$ be a rotation matrix, given as a function of time
- since $I = R(t)R^T(t)$, taking the time derivative of both sides yields

$$0 = d[R(t)R^T(t)]/dt = dR(t)/dt\, R^T(t) + R(t)\, dR^T(t)/dt$$

$$= dR(t)/dt\, R^T(t) + [dR(t)/dt\, R^T(t)]^T$$

A skew-symmetric $\iff A^T = -A$

thus $dR(t)/dt\, R^T(t) = S(t)$ is a skew-symmetric matrix

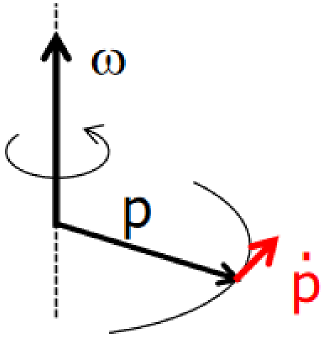


- let $p(t) = R(t)p'$ a vector (with constant norm) rotated over time
- comparing

$$dp(t)/dt = dR(t)/dt\, p' = S(t)R(t)\, p' = S(t)\, p(t)$$

$$dp(t)/dt = \omega(t) \times p(t) = S(\omega(t))\, p(t)$$

we get $S = S(\omega)$



$$\dot{R} = S(\omega)\, R \iff S(\omega) = \dot{R}\, R^T$$

Example

Time derivative of an elementary rotation matrix

$$R_X(\phi(t)) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi(t) & -\sin\phi(t) \\ 0 & \sin\phi(t) & \cos\phi(t) \end{bmatrix}$$

$$\dot{R}_X(\phi)\, R^T{}_X(\phi) = \dot{\phi} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin\phi & -\cos\phi \\ 0 & \cos\phi & -\sin\phi \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\dot{\phi} \\ 0 & \dot{\phi} & 0 \end{bmatrix} = S(\omega) \qquad \boldsymbol{S} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix},$$

$$\omega = \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix}$$

S(t): physical interpretation.

$$\dot{\boldsymbol{p}}(t) = \boldsymbol{\omega}(t) \times \boldsymbol{R}(t)\boldsymbol{p}'.$$

Therefore, the matrix operator ***S***(*t*) describes the vector product between the vector ***ω*** and the vector ***R***(*t*)***p***^{I}.

The matrix ***S***(*t*) is so that its symmetric elements with respect to the main diagonal represent the components of the vector ***ω***(*t*) = [$\omega_x \omega_y \omega_z$]^T in the form:

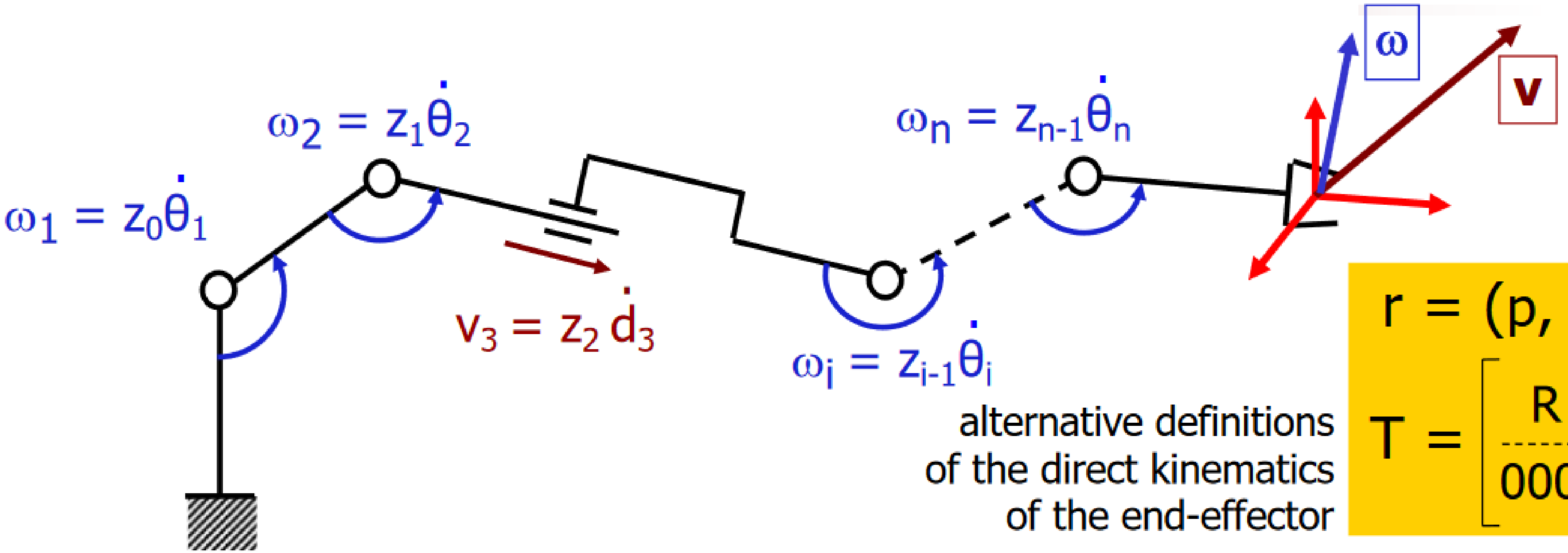
$$\boldsymbol{S} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix},$$

$$\boldsymbol{S}(t) = \boldsymbol{S}(\boldsymbol{\omega}(t)).$$

We can rewrite

$$\dot{\boldsymbol{R}}(t) = \boldsymbol{S}(t)\boldsymbol{R}(t) \quad \rightarrow \quad \dot{\boldsymbol{R}} = \boldsymbol{S}(\boldsymbol{\omega})\boldsymbol{R}.$$

Linear and angular velocity of the robot end-effector



alternative definitions of the direct kinematics of the end-effector

$$r = (p, \phi)$$

$$T = \begin{bmatrix} R & p \\ 000 & 1 \end{bmatrix}$$

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Robot Jacobian matrices

- analytical Jacobian (obtained by time differentiation)

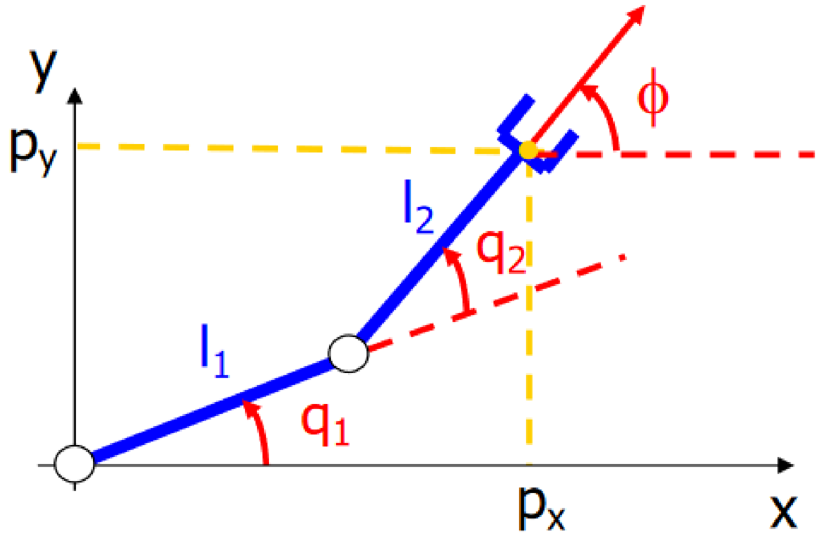
$$r = \begin{pmatrix} p \\ \phi \end{pmatrix} = f_r(q) \quad \Rightarrow \quad \dot{r} = \begin{pmatrix} \dot{p} \\ \dot{\phi} \end{pmatrix} = \frac{\partial f_r(q)}{\partial q} \dot{q} = J_r(q)\, \dot{q}$$

- geometric Jacobian (no derivatives)

$$\begin{pmatrix} v \\ \omega \end{pmatrix} = \begin{pmatrix} \dot{p} \\ \omega \end{pmatrix} = \begin{pmatrix} J_L(q) \\ J_A(q) \end{pmatrix} \dot{q} = J(q)\, \dot{q}$$

- in both cases, the Jacobian matrix depends on the (current) configuration of the robot

Analytical Jacobian of planar 2R arm



direct kinematics

$$r \left\{ \begin{array}{l} p_x = l_1 c_1 + l_2 c_{12} \\ p_y = l_1 s_1 + l_2 s_{12} \\ \hline \phi = q_1 + q_2 \end{array} \right.$$

$$\dot{p}_x = - l_1 s_1 \dot{q}_1 - l_2 s_{12} (\dot{q}_1 + \dot{q}_2)$$

$$\dot{p}_y = l_1 c_1 \dot{q}_1 + l_2 c_{12} (\dot{q}_1 + \dot{q}_2)$$

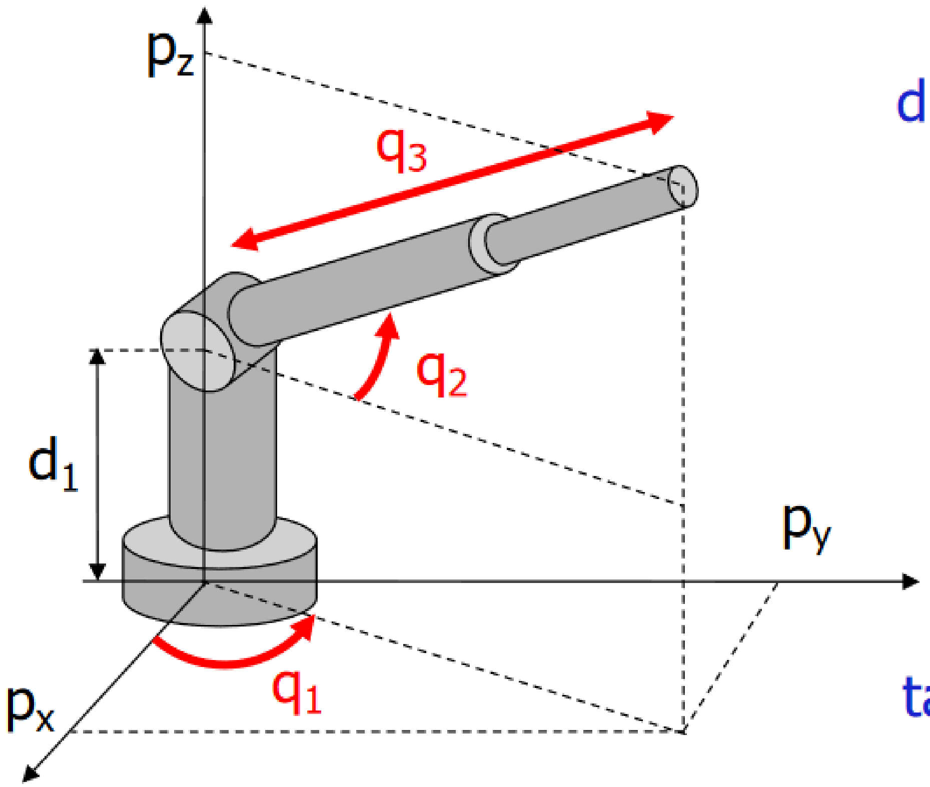
$$\dot{\phi} = \omega_z = \dot{q}_1 + \dot{q}_2$$

$$\Rightarrow \quad J_r(q) = \begin{pmatrix} - l_1 s_1 - l_2 s_{12} & - l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \\ \hline 1 & 1 \end{pmatrix}$$

given r, this is a 3 x 2 matrix

here, all rotations occur around the same fixed axis z (normal to the plane of motion)

Analytical Jacobian of polar robot



direct kinematics (here, r = p)

$$\left.\begin{aligned} p_x &= q_3\, c_2\, c_1 \\ p_y &= q_3\, c_2\, s_1 \\ p_z &= d_1 + q_3\, s_2 \end{aligned}\right\} f_r(q)$$

taking the time derivative

$$v = \dot{p} = \underbrace{\begin{pmatrix} -q_3c_2s_1 & -q_3s_2c_1 & c_2c_1 \\ q_3c_2c_1 & -q_3s_2s_1 & c_2s_1 \\ 0 & q_3c_2 & s_2 \end{pmatrix}}_{\frac{\partial f_r(q)}{\partial q}} \dot{q} = J_r(q)\, \dot{q}$$

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https://en.wikipedia.org/wiki/Superposition_principle

Superposition principle

This article is about the superposition principle in linear systems. For other uses, see Superposition (disambiguation).

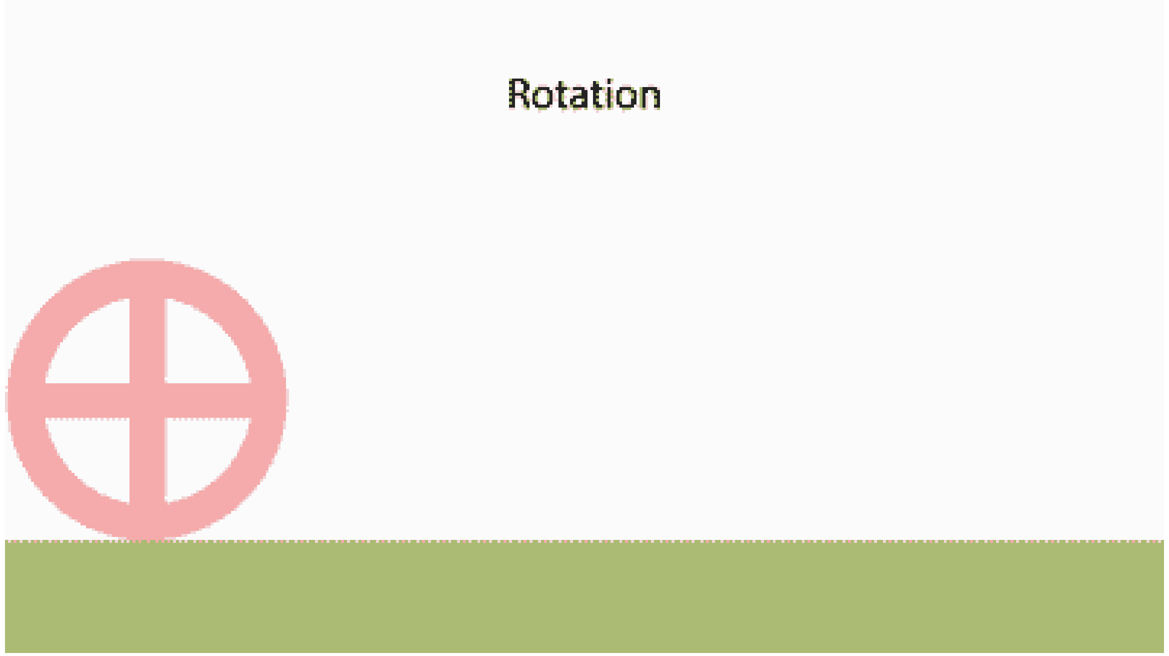
The **superposition principle**,[1] also known as **superposition property**, states that, for all linear systems, the net response caused by two or more stimuli is the sum of the responses that would have been caused by each stimulus individually. So that if input *A* produces response *X* and input *B* produces response *Y* then input (*A* + *B*) produces response (*X* + *Y*).

A function $F(x)$ that satisfies the superposition principle is called a linear function. Superposition can be defined by two simpler properties; additivity and homogeneity

$$F(x_1 + x_2) = F(x_1) + F(x_2)$$ **Additivity**

$$F(ax) = aF(x)$$ **Homogeneity**

for scalar a.



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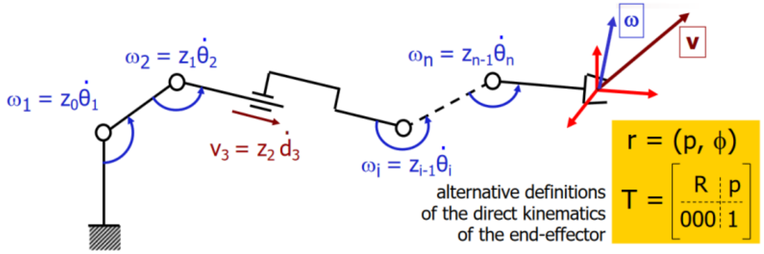
Geometric Jacobian (Generalizing of *n-dof*)

Linear and angular velocity of the robot end-effector



always a 6 x n matrix

end-effector instantaneous velocity

$$\begin{pmatrix} v_E \\ \omega_E \end{pmatrix} = \begin{pmatrix} J_L(q) \\ J_A(q) \end{pmatrix} \dot{q} = \begin{pmatrix} J_{L1}(q) & \cdots & J_{Ln}(q) \\ J_{A1}(q) & \cdots & J_{An}(q) \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{pmatrix}$$

superposition of effects

$$v_E = J_{L1}(q)\,\dot{q}_1 + \ldots + J_{Ln}(q)\,\dot{q}_n \qquad \omega_E = J_{A1}(q)\,\dot{q}_1 + \ldots + J_{An}(q)\,\dot{q}_n$$

contribution to the linear e-e velocity due to $\dot{q}_1$

contribution to the angular e-e velocity due to $\dot{q}_1$

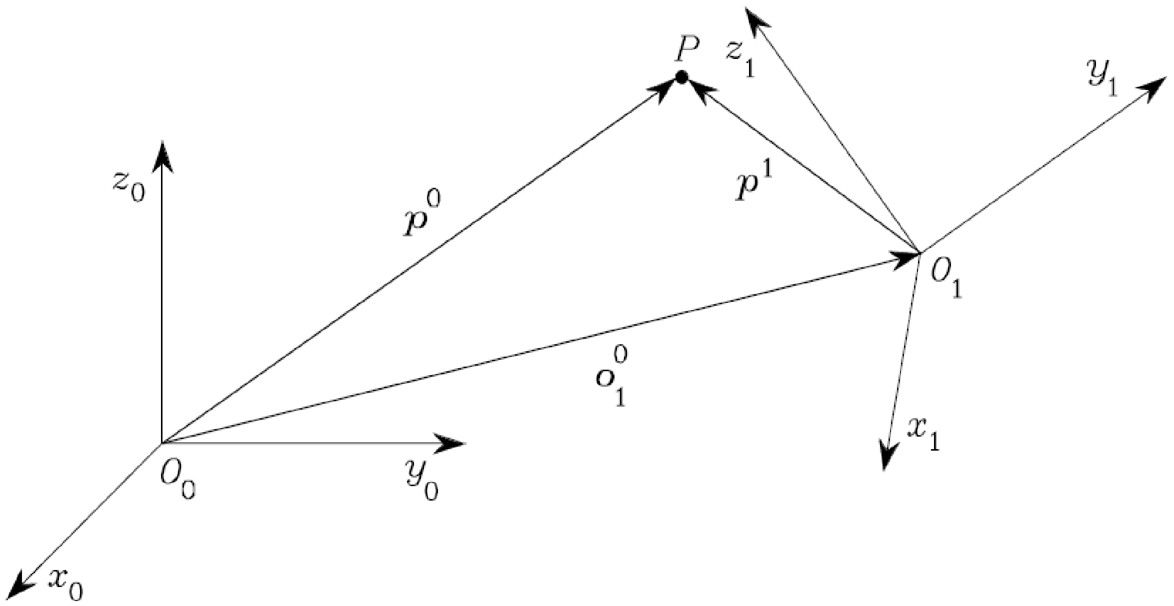
linear and angular velocity belong to (linear) vector spaces in R^3

Velocity composition rule.

Generic approach for two frames

Let s consider the coordinate transformation of a point *P* from Frame 1 to Frame 0 given by

$$p^0 = o_1^0 + R_1^0 p^1.$$



Differentiating with respect to time and using $\dot{R} = S(\omega)R.$ 'es:

$$\dot{p}^0 = \dot{o}_1^0 + R_1^0 \dot{p}^1 + \dot{R}_1^0 p^1;$$

$$\dot{p}^0 = \dot{o}_1^0 + R_1^0 \dot{p}^1 + S(\omega_1^0) R_1^0 p^1.$$

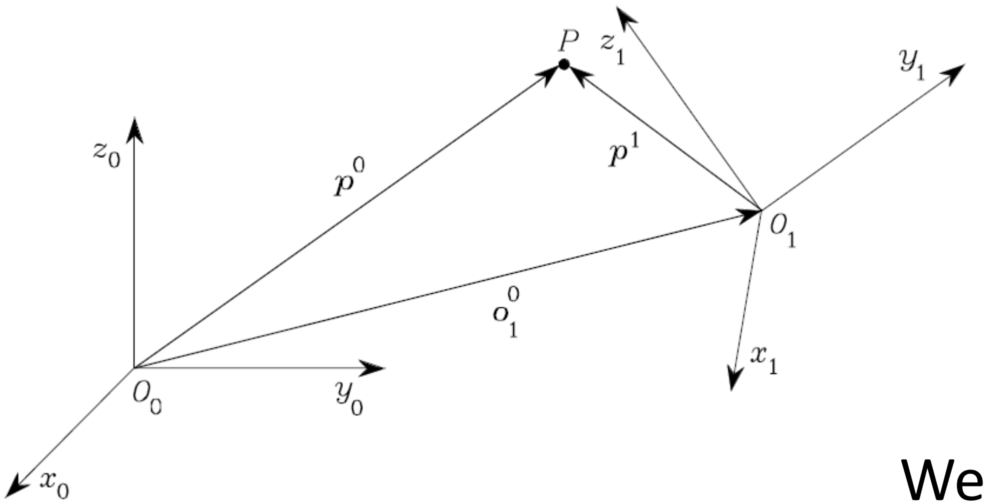
Expressing $R_1^0 p^1$ by r_1^0

$$\boxed{\dot{p}^0 = \dot{o}_1^0 + R_1^0 \dot{p}^1 + \omega_1^0 \times r_1^0}$$

Velocity composition rule

$$\dot{\boldsymbol{p}}^0 = \dot{\boldsymbol{o}}_1^0 + \boldsymbol{R}_1^0 \dot{\boldsymbol{p}}^1 + \boldsymbol{\omega}_1^0 \times \boldsymbol{r}_1^0$$

1 **2** **3**



We set :

$\boldsymbol{R}_1^0 \boldsymbol{p}^1$ by $\boldsymbol{r}_1^0$

1- Linear Velocity of the origin O_1 of $x_1y_1z_1$ respect to $x_0y_0z_0$

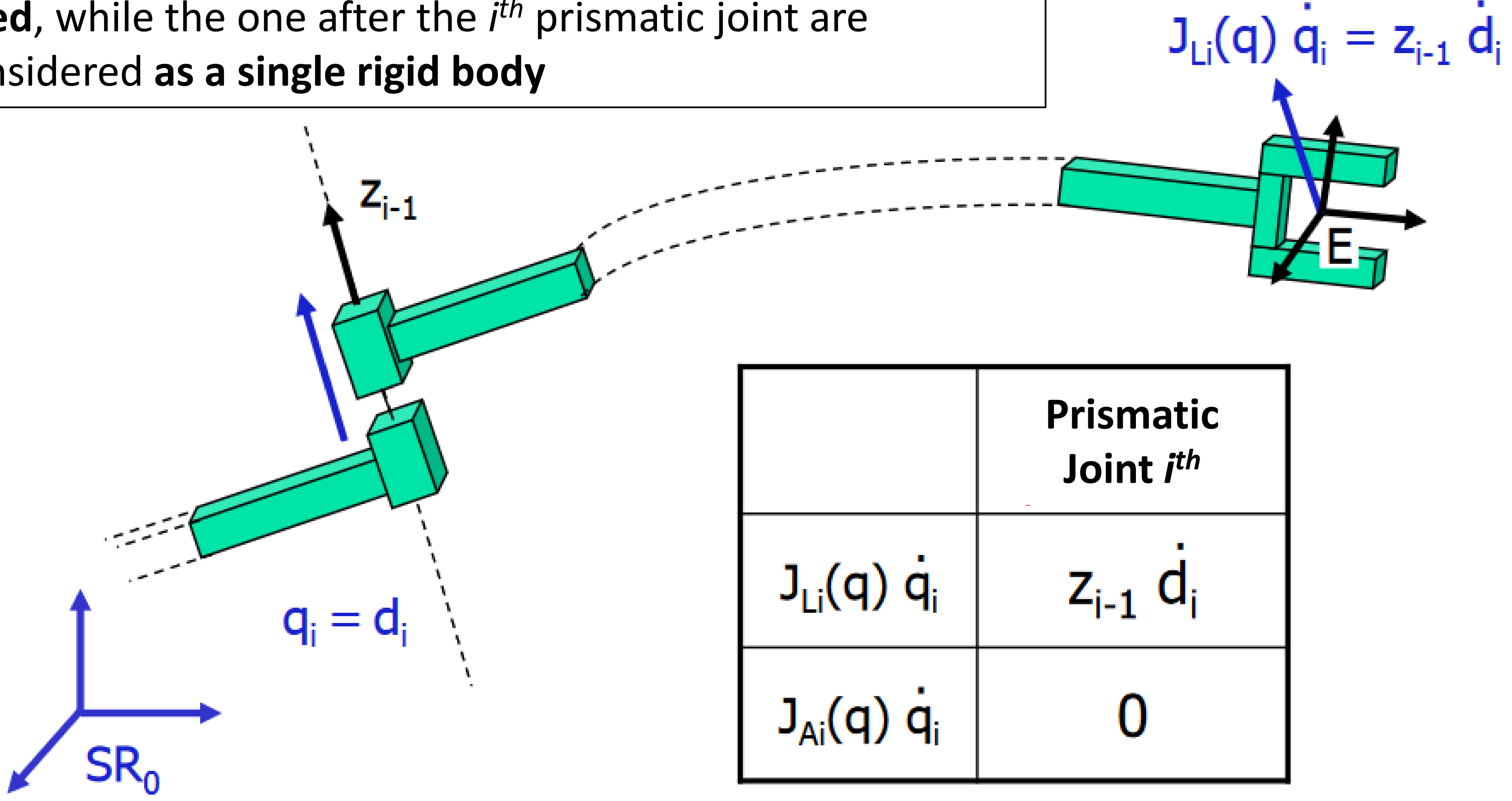
2- Linear Velocity of the vector *P* respect to $x_1y_1z_1$ (**=0** because P^1 is fixed respect to $x_1y_1z_1$) $\dot{\boldsymbol{p}}^1 = \boldsymbol{0}$.

3- Linear Velocity of the point *P* respect to $x_0y_0z_0$

$$\dot{\boldsymbol{p}}^0 = \dot{\boldsymbol{o}}_1^0 + \boldsymbol{\omega}_1^0 \times \boldsymbol{r}_1^0$$

Geometric Jacobian Computation

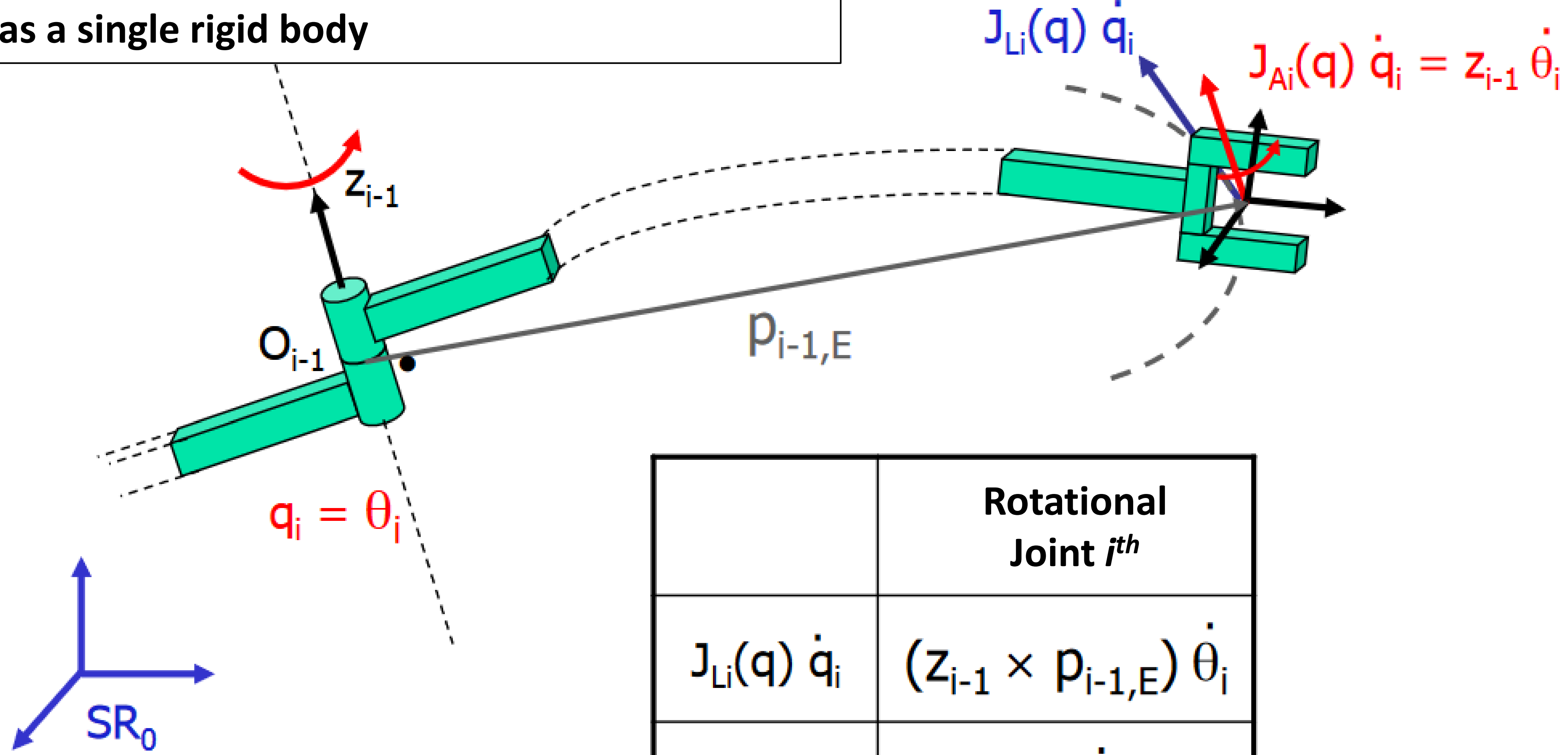
The joints before the i^{th} prismatic joint are considered **fixed**, while the one after the i^{th} prismatic joint are considered **as a single rigid body**



	Prismatic Joint i^{th}
$J_{Li}(q)\,\dot{q}_i$	$z_{i-1}\,\dot{d}_i$
$J_{Ai}(q)\,\dot{q}_i$	0

Geometric Jacobian Computation

The joints before the i^{th} rotational joint are considered **fixed**, while the one after the i^{th} rotational joint are considered **as a single rigid body**



	Rotational Joint i^{th}
$J_{Li}(q)\,\dot{q}_i$	$(z_{i-1} \times p_{i-1,E})\,\dot{\theta}_i$
$J_{Ai}(q)\,\dot{q}_i$	$z_{i-1}\,\dot{\theta}_i$

Expression of geometric Jacobian

$$\left(\begin{pmatrix}\dot{p}_{0,E}\\ \omega_E\end{pmatrix}=\right)\begin{pmatrix}v_E\\ \omega_E\end{pmatrix}=\begin{pmatrix}J_L(q)\\ J_A(q)\end{pmatrix}\dot{q}=\begin{pmatrix}J_{L1}(q) & \dots & J_{Ln}(q)\\ J_{A1}(q) & \dots & J_{An}(q)\end{pmatrix}\begin{pmatrix}\dot{q}_1\\ \vdots\\ \dot{q}_n\end{pmatrix}$$

	prismatic i-th joint	revolute i-th joint
$J_{Li}(q)$	z_{i-1}	$z_{i-1} \times p_{i-1,E}$
$J_{Ai}(q)$	0	z_{i-1}

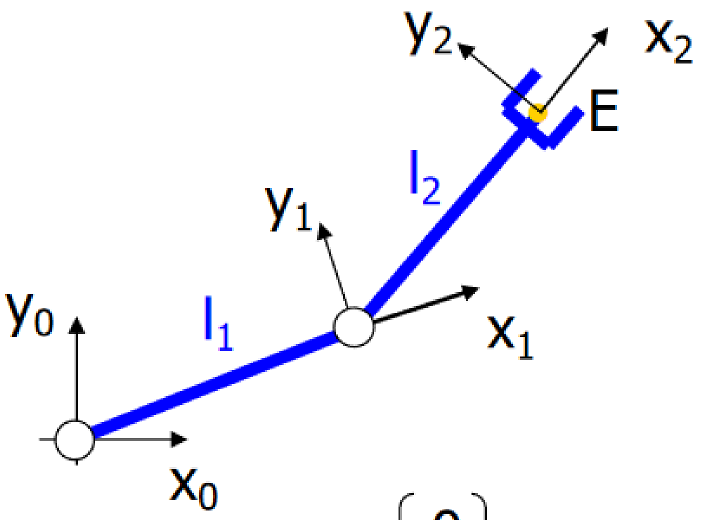
this can be also computed as $= \frac{\partial p_{0,E}}{\partial q_i}$

$$z_{i-1} = {}^0R_1(q_1)\dots{}^{i-2}R_{i-1}(q_{i-1})\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$p_{i-1,E} = p_{0,E}(q_1,\dots,q_n) - p_{0,i-1}(q_1,\dots,q_{i-1})$$

all vectors should be expressed in the same reference frame (here, the base frame RF_0)

Example: planar 2R arm



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joint	α_i	d_i	a_i	θ_i
1	0	0	l_1	q_1
2	0	0	l_2	q_2

$$z_0 = z_1 = z_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

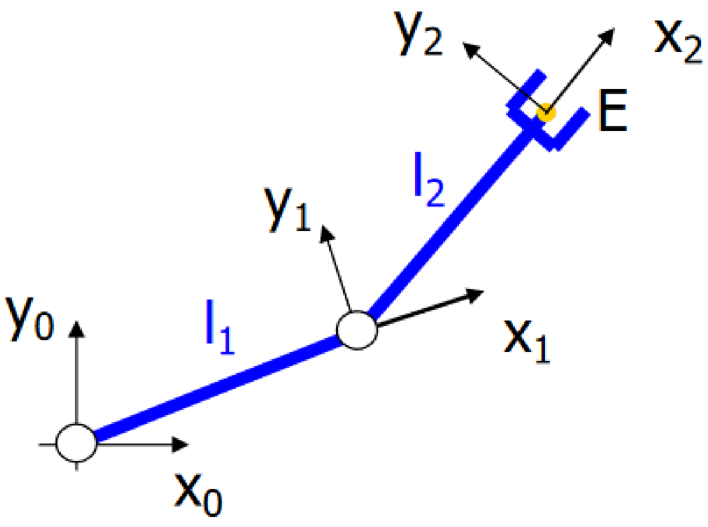
$$^0A_1 = \begin{pmatrix} c_1 & -s_1 & 0 & l_1 c_1 \\ s_1 & c_1 & 0 & l_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \leftarrow p_{0,1}$$

$$p_{1,E} = p_{0,E} - p_{0,1}$$

$$J = \begin{bmatrix} z_0 \times p_{0,E} & z_1 \times p_{1,E} \\ z_0 & z_1 \end{bmatrix}$$

$$^0A_2 = \begin{pmatrix} c_{12} & -s_{12} & 0 & l_1 c_1 + l_2 c_{12} \\ s_{12} & c_{12} & 0 & l_1 s_1 + l_2 s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \leftarrow p_{0,E}$$

Geometric Jacobian of planar 2R arm



$$J = \begin{pmatrix} z_0 \times p_{0,E} & z_1 \times p_{1,E} \\ z_0 & z_1 \end{pmatrix}$$

$$= \begin{pmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}$$

compare rows 1, 2, and 6 with the analytical Jacobian

note: the Jacobian is here a 6×2 matrix, thus its maximum rank is 2

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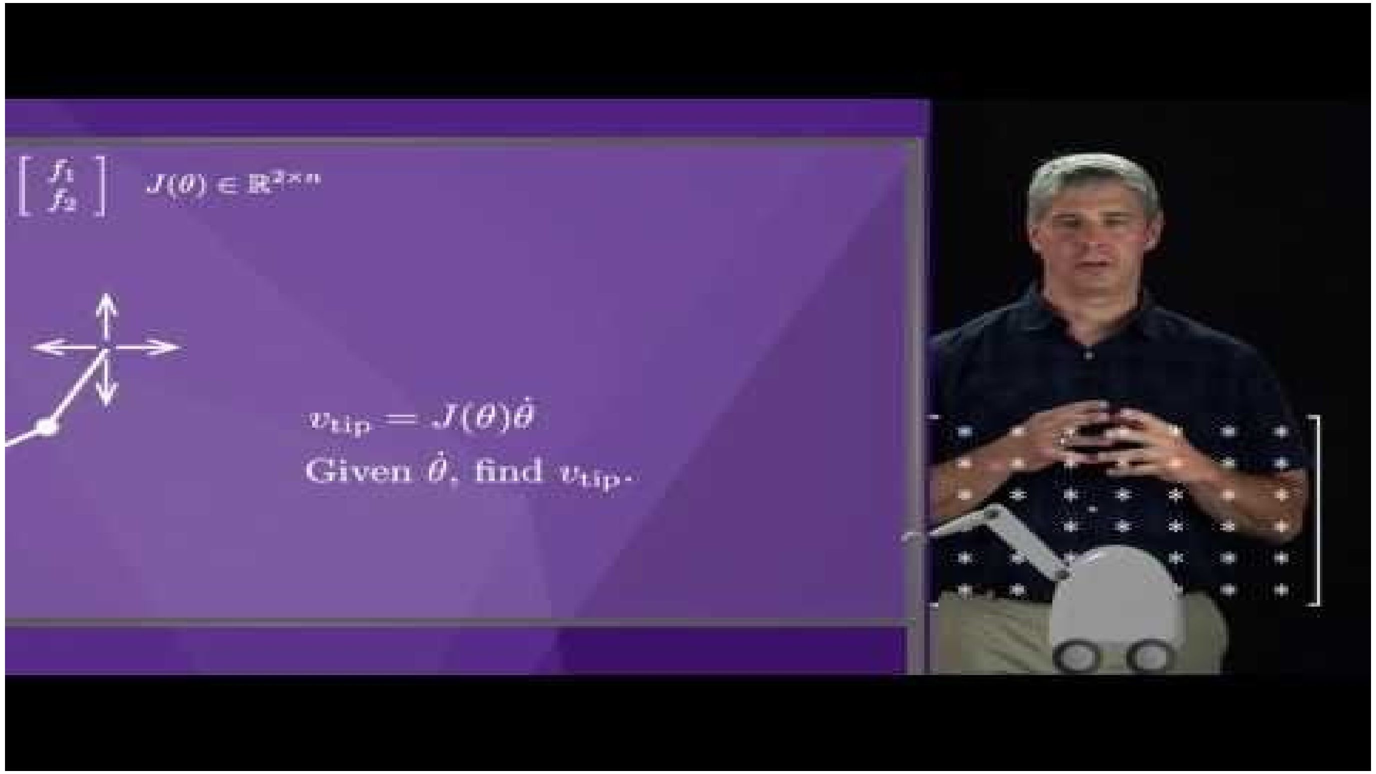
at most 2 components of the linear/angular end-effector velocity can be independently assigned

Video from Kevin Lynch Instructional

https://www.youtube.com/watch?v=vjJgTvnQpBs



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Acceleration relations (and beyond...)

Higher-order differential kinematics

- differential relations between motion in the joint space and motion in the task space can be established at the second order, third order, ...
- the analytical Jacobian always "weights" the highest-order derivative

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velocity $\dot{r} = J_r(q)\,\dot{q}$

matrix function $N_2(q,\dot{q})$

acceleration $\ddot{r} = J_r(q)\,\ddot{q} + \dot{J}_r(q)\,\dot{q}$

jerk $\dddot{r} = J_r(q)\,\dddot{q} + 2\,\dot{J}_r(q)\,\ddot{q} + \ddot{J}_r(q)\,\dot{q}$

snap $\ddddot{r} = J_r(q)\,\ddddot{q} + \dots$

matrix function $N_3(q,\dot{q},\ddot{q})$

- the same holds true also for the geometric Jacobian $J(q)$

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The end!