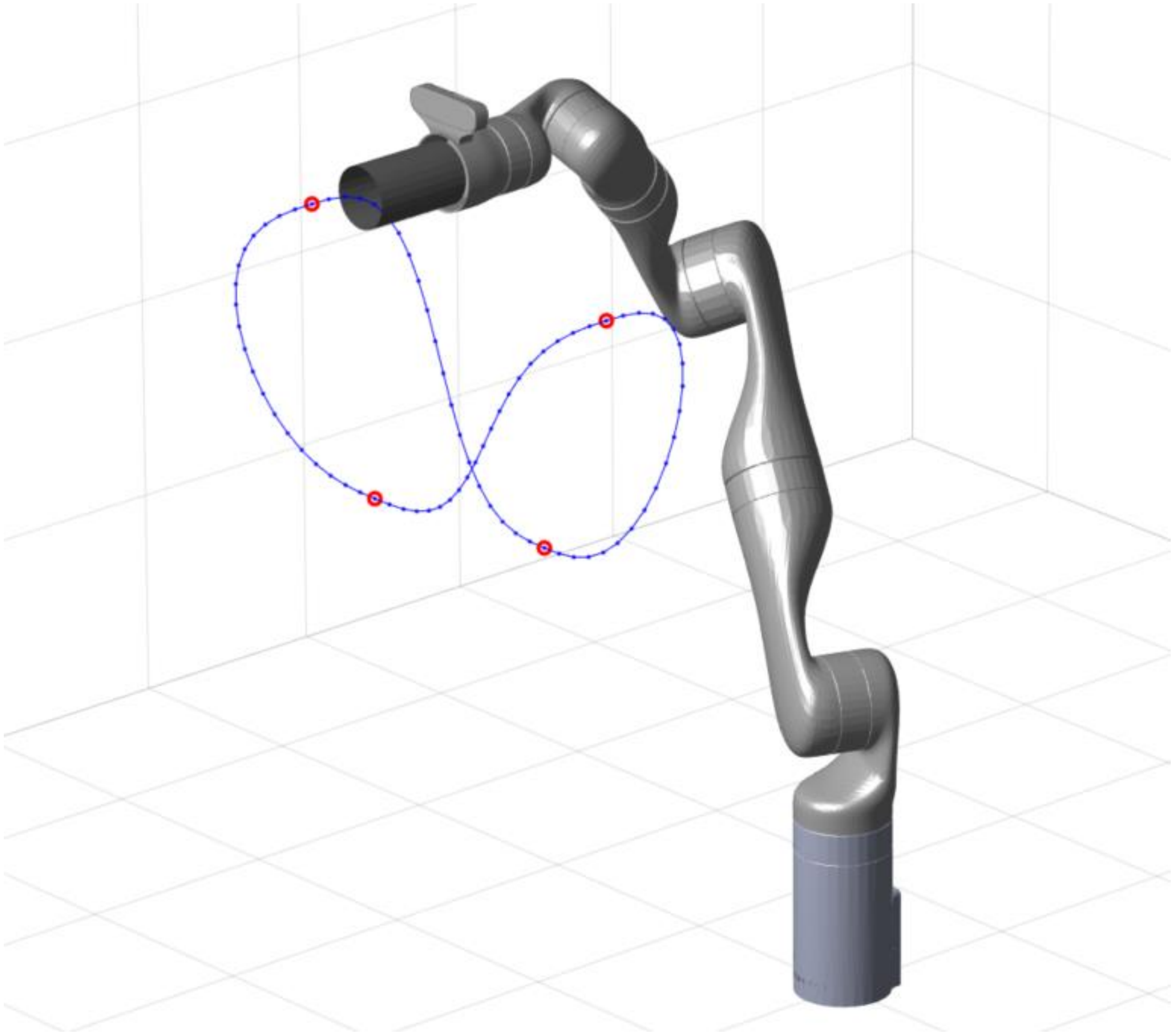
Motion-Trajectory Planning in Cartesian Space



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Recalling trajectories in the joints space (1)

- after choosing a path, the trajectory definition is completed by the choice of a timing law

$$p = p(s) \quad \Rightarrow s = s(t) \qquad \text{(Cartesian space)}$$

$$q = q(\lambda) \quad \Rightarrow \lambda = \lambda(t) \qquad \text{(joint space)}$$

  - if $s(t) = t$, path parameterization is the natural one given by time

- the timing law
  - is chosen based on task specifications (stop in a point, move at constant velocity, and so on)
  - may consider optimality criteria (min transfer time, min energy,…)
  - constraints are imposed by actuator capabilities (max torque, max velocity,…) and/or by the task (e.g., max acceleration on payload)

note: on parameterized paths, a space-time decomposition takes place

e.g., in Cartesian space $\quad \dot{p}(t) = \frac{dp}{ds}\dot{s} \qquad \ddot{p}(t) = \frac{dp}{ds}\ddot{s} + \frac{d^2p}{ds^2}\dot{s}^2$

Cartesian vs. joint trajectory planning (2)

- planning in Cartesian space
  - allows a more direct visualization of the generated path
  - obstacle avoidance, lack of "wandering"
- planning in joint space
  - does not need on-line kinematic inversion
- issues in kinematic inversion
  - $\dot{q}$ e $\ddot{q}$ (or higher-order derivatives) may also be needed
    - Cartesian task specifications involve the geometric path, but also bounds on the associated timing law
  - for redundant robots, choice among ∞^{n-m} inverse solutions, based on optimality criteria or additional auxiliary tasks
  - off-line planning in advance is not always feasible
    - e.g., when interaction with the environment occurs or sensor-based motion is needed

Trajectory planning in joint space (3)

- $q = q(t)$ in time or $q = q(\lambda)$ in space (then with $\lambda = \lambda(t)$)
- it is sufficient to work component-wise (q_i in vector q)
- an implicit definition of the trajectory, by solving a problem with specified boundary conditions in a given class of functions
- typical classes: polynomials (cubic, quintic,...), (co)sinusoids, clothoids, ...

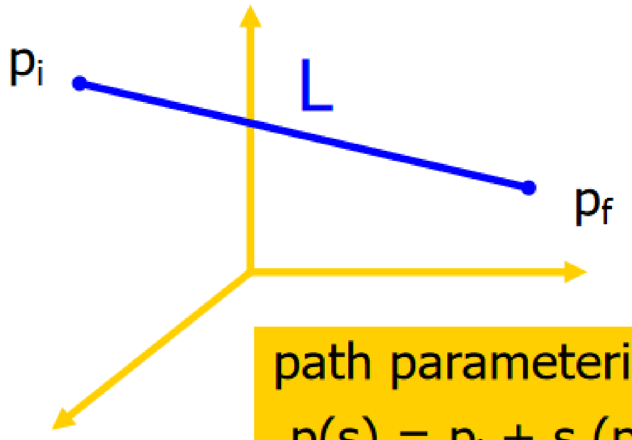
- imposed conditions
  - passage through points = interpolation
  - initial, final, intermediate velocity (or geometric tangent for paths)
  - initial, final acceleration (or geometric curvature)
  - continuity up to the k-th order time (or space) derivative: class $\mathbf{C}^k$

many of the following methods and remarks can be directly applied also to Cartesian trajectory planning (and vice versa)!

Trajectories in Cartesian space

- in general, the trajectory planning methods proposed in the joint space can be applied also in the Cartesian space
  - consider independently each component of the task vector (i.e., a position or an angle of a minimal representation of orientation)
- however, when planning a trajectory for the three orientation angles, the resulting global motion cannot be intuitively visualized in advance
- if possible, we still prefer to plan Cartesian trajectories separately for position and orientation
- the number of knots to be interpolated in the Cartesian space is typically low (e.g., 2 knots for a PTP motion, 3 if a "via point" is added) ⇒ use simple interpolating paths, such as straight lines, arc of circles, …

Planning a linear Cartesian path
(position only)



GIVEN
p_i, p_f, v_{max}, a_{max}
v_i, v_f (typically = 0)

path parameterization

$$p(s) = p_i + s\,(p_f - p_i)$$

$$L = \| p_f - p_i \|$$

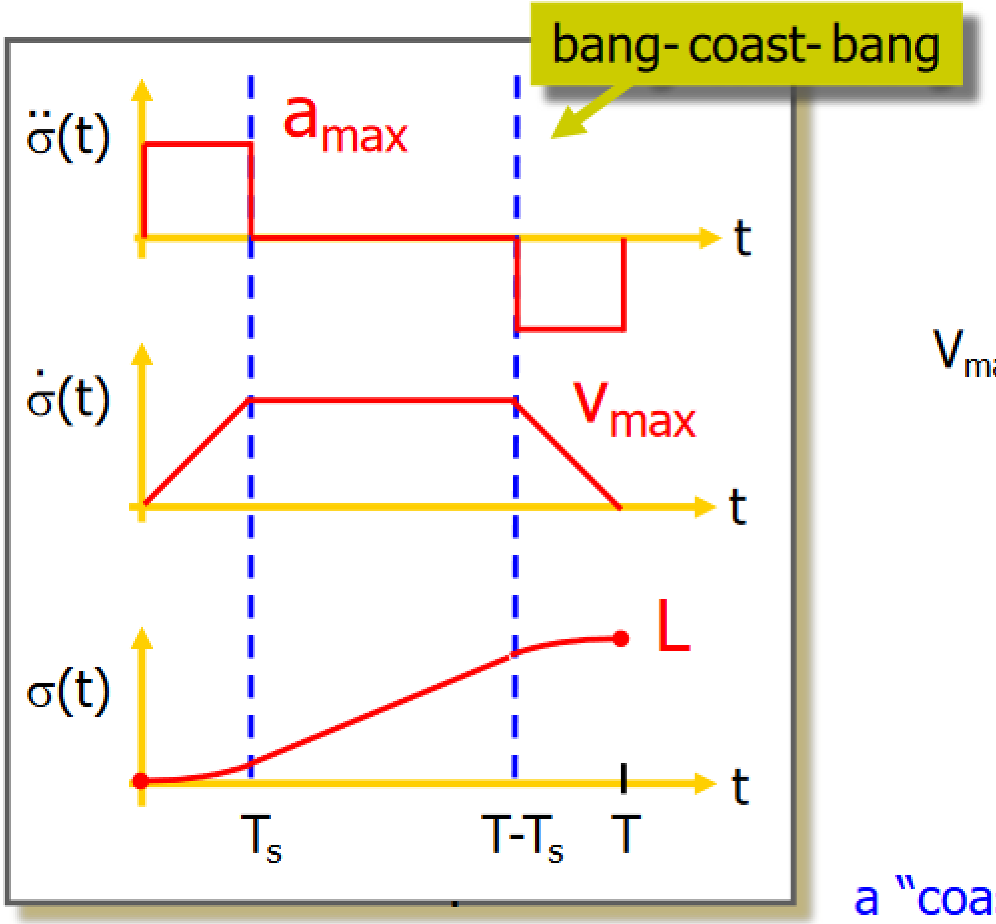
$$\frac{p_f - p_i}{\| p_f - p_i \|} = \text{unit vector of directional cosines of the line}$$

$s \in [0,1]$ ← setting $s = \sigma/L$, $\sigma \in [0,L]$ is the arc length (gives the current length of the path)

$$\dot{p}(s) = \frac{dp}{ds}\,\dot{s} = (p_f - p_i)\,\dot{s} = \frac{p_f - p_i}{L}\,\dot{\sigma}$$

$$\ddot{p}(s) = \cancel{\frac{d^2p}{ds^2}}\,\dot{s}^2 + \frac{dp}{ds}\,\ddot{s} = (p_f - p_i)\,\ddot{s} = \frac{p_f - p_i}{L}\,\ddot{\sigma}$$

Timing law with trapezoidal speed - 1



given*: L, v_{max}, a_{max}
find: T_s, T

$$V_{max}(T - T_s) = L$$

= area of the speed profile

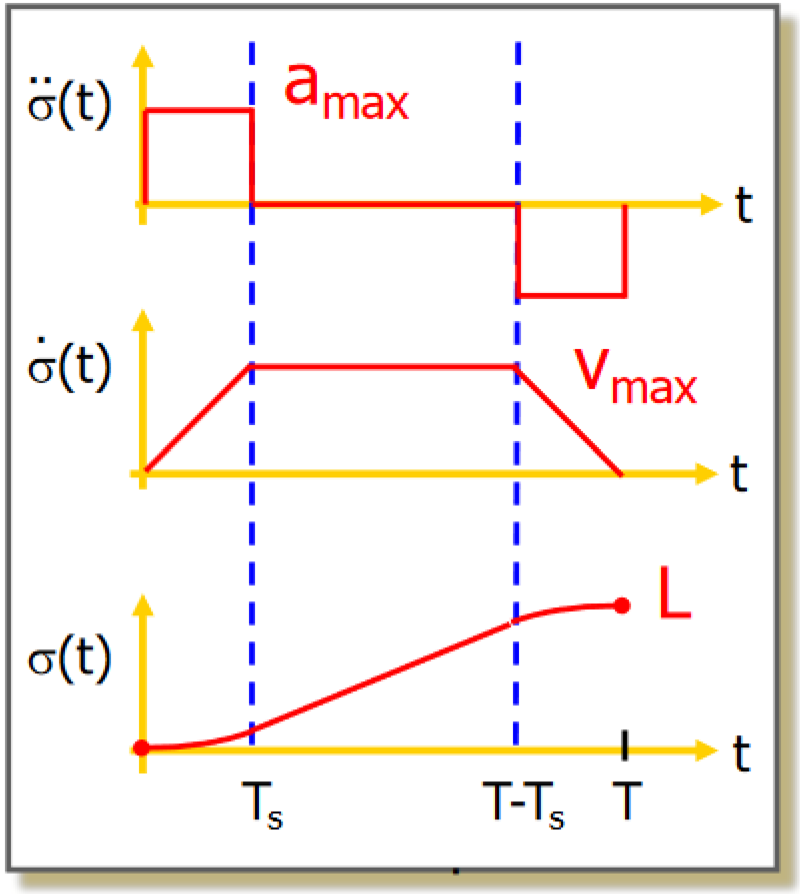
$$T_s = \frac{v_{max}}{a_{max}}$$

$$T = \frac{L\, a_{max} + v_{max}^2}{a_{max} v_{max}}$$

a "coast" phase exists iff: $L > v_{max}^2/a_{max}$

* = other input data combinations are possible

Timing law with trapezoidal speed - 2



$$\sigma(t) = \begin{cases} a_{max}\, t^2/2 & t \in [0, T_s] \\ v_{max}\, t - \dfrac{v_{max}^2}{2\, a_{max}} & t \in [T_s, T-T_s] \\ -\, a_{max}\, (t-T)^2/2 + v_{max}\, T - \dfrac{v_{max}^2}{a_{max}} & t \in [T-T_s, T] \end{cases}$$

can be used also in the joint space!

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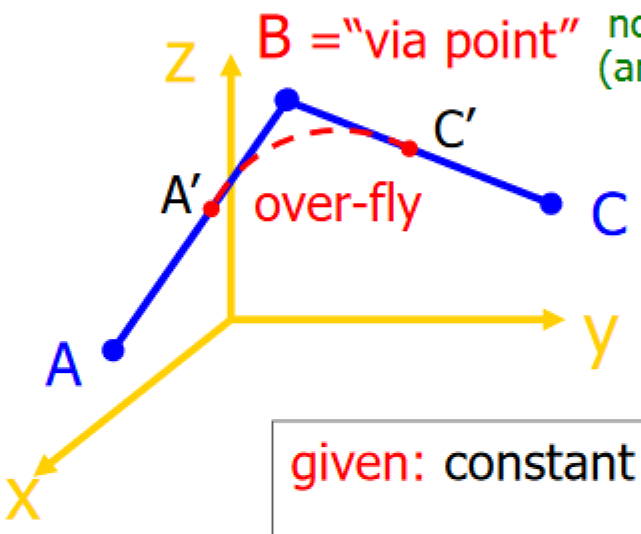
Concatenation of linear paths



no need to pass (and stop!) there

$$\frac{B - A}{\|B - A\|} = K_{AB}$$

$$\frac{C - B}{\|C - B\|} = K_{BC}$$

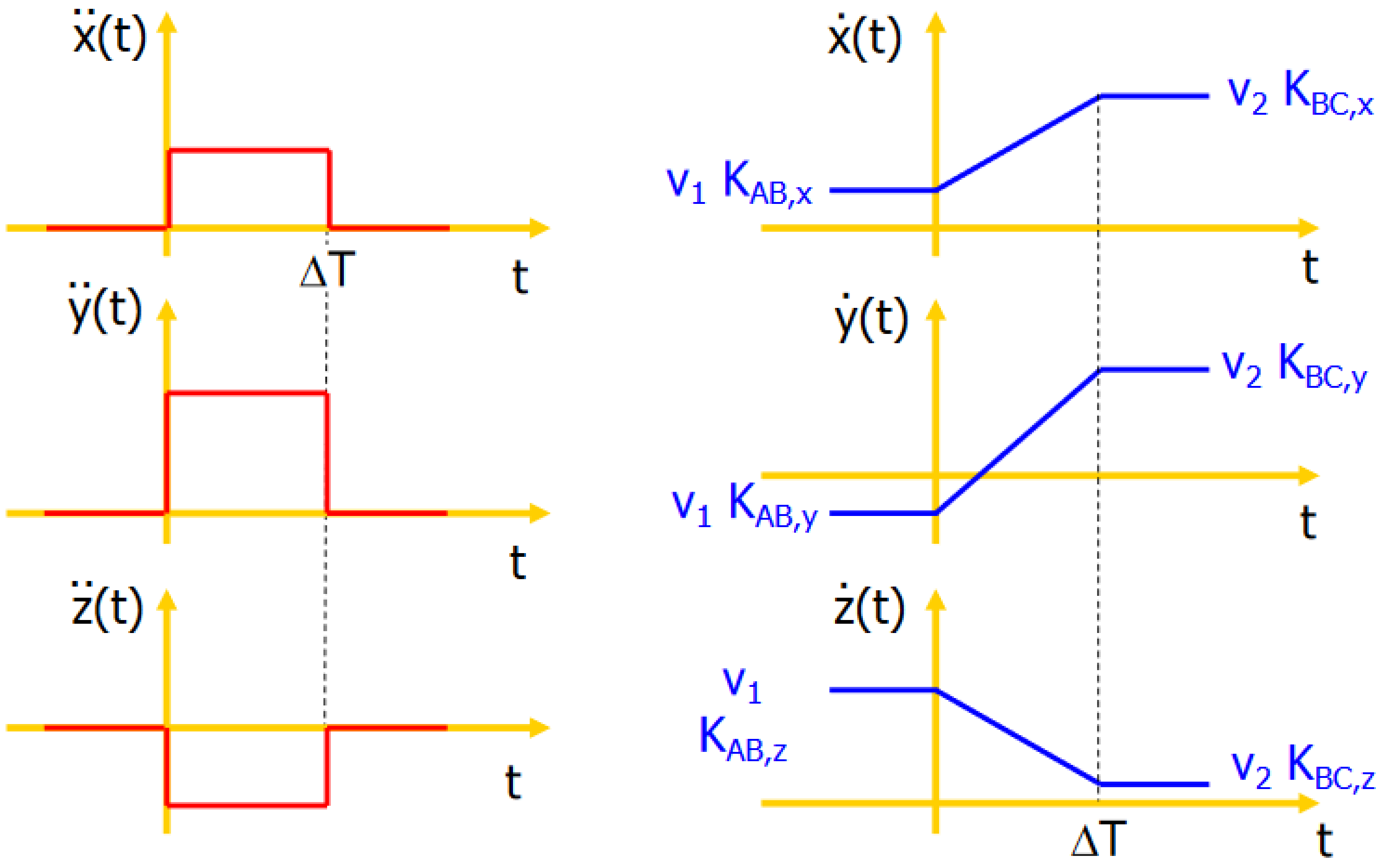
unit vectors of direction cosines

given: constant speeds v_1 on linear path AB
v_2 on linear path BC
desired transition: with constant acceleration for a time ΔT

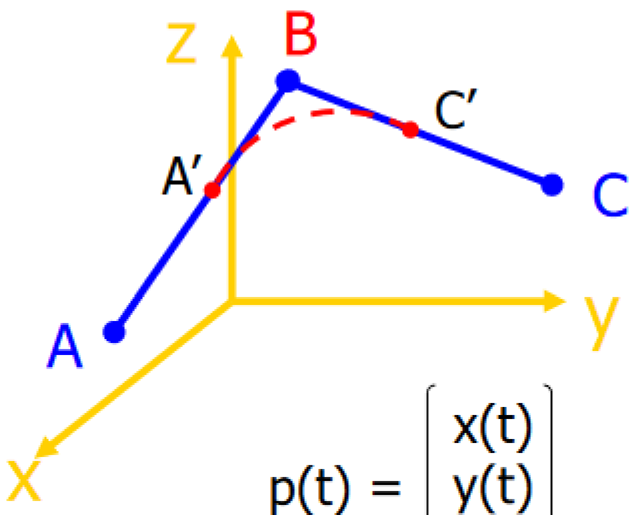
$$p(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} \quad t \in [0, \Delta T] \text{ (transition starts at } t = 0)$$

note: during over-fly, the path remains always in the plane specified by the two lines intersecting at B (in essence, it is a planar problem)

Time profiles on components



Timing law during transition



$$p(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

$$\frac{B - A}{\|B - A\|} = K_{AB}$$

$$\frac{C - B}{\|C - B\|} = K_{BC}$$

unit vectors of direction cosines

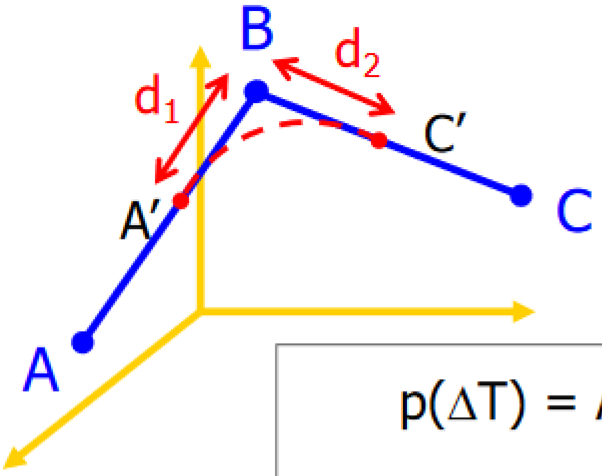
$t \in [0, \Delta T]$ (transition starts at $t = 0$)

$$\ddot{p}(t) = (v_2 K_{BC} - v_1 K_{AB})/\Delta T \quad \xrightarrow{\int} \quad \dot{p}(t) = v_1 K_{AB} + (v_2 K_{BC} - v_1 K_{AB})\, t/\Delta T$$

$$\xrightarrow{\int} \quad p(t) = A' + v_1 K_{AB}\, t + (v_2 K_{BC} - v_1 K_{AB})\, t^2/(2\Delta T)$$

thus, we obtain a parabolic blending

Solution
(various options)



$$B - A' = d_1 K_{AB}$$
$$C' - B = d_2 K_{BC} \quad (1)$$

$$p(t) = A' + v_1 K_{AB} t + (v_2 K_{BC} - v_1 K_{AB}) t^2/(2\Delta T)$$

$$p(\Delta T) = A' + (\Delta T/2)(v_1 K_{AB} + v_2 K_{BC}) = C'$$

$$\Rightarrow -B + A' + (\Delta T/2)(v_1 K_{AB} + v_2 K_{BC}) = C' - B$$

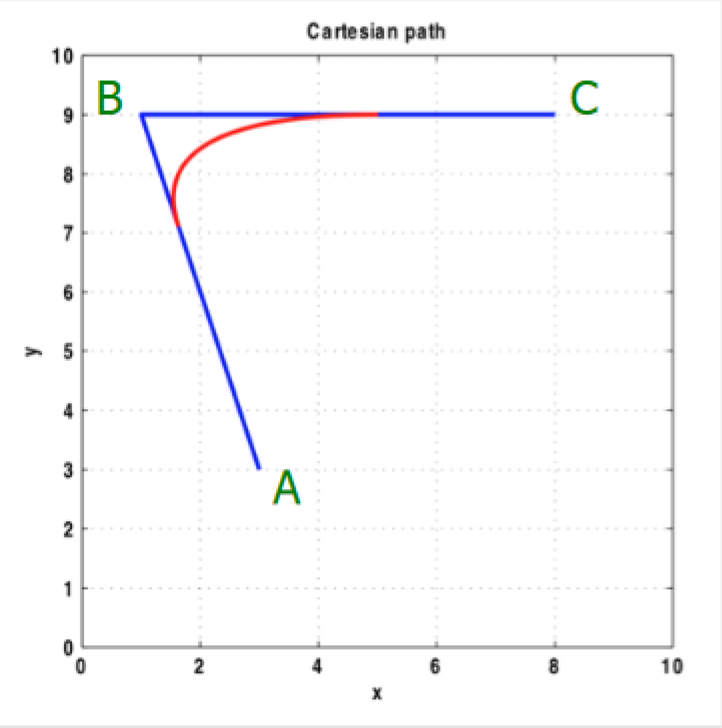
$$\overset{(1)}{\Rightarrow} d_1 K_{AB} + d_2 K_{BC} = (\Delta T/2)(v_1 K_{AB} + v_2 K_{BC})$$

$$\Rightarrow \boxed{d_1 = v_1 \Delta T/2} \quad \boxed{d_2 = v_2 \Delta T/2}$$

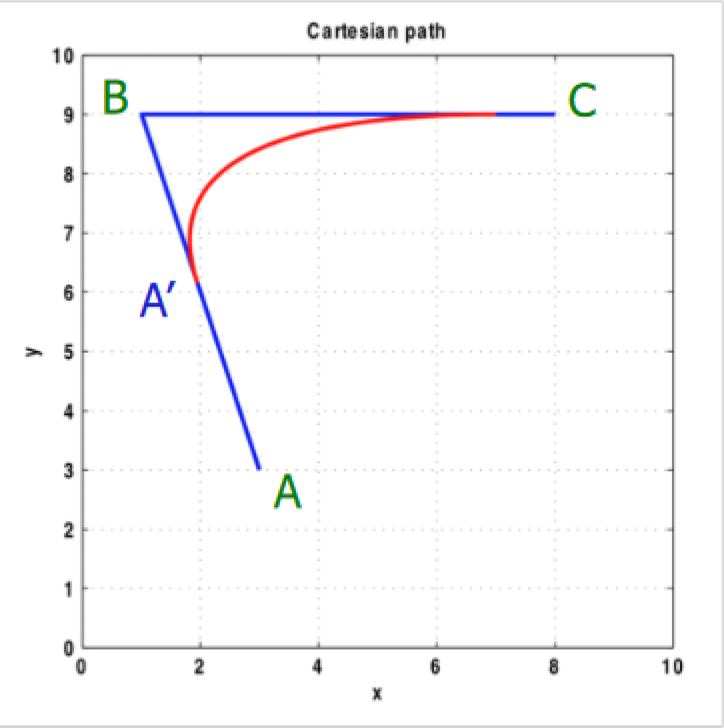
by choosing, e.g., d_1 (namely A′) → $\Delta T = 2d_1/v_1 \Rightarrow d_2 = d_1 v_2/v_1$

A numerical example

- transition from $A=(3,3)$ to $C=(8,9)$ via $B=(1,9)$, with speed from $v_1=1$ to $v_2=2$
- exploiting two options for solution (resulting in different paths!)
  - assign transition time: $\Delta T=4$ (we re-center it here for $t \in [-\Delta T/2, \Delta T/2]$)
  - assign distance from B for departing: $d_1=3$ (assign d_2 for landing is handled similarly)



$\Delta T=4$



$d_1=3$

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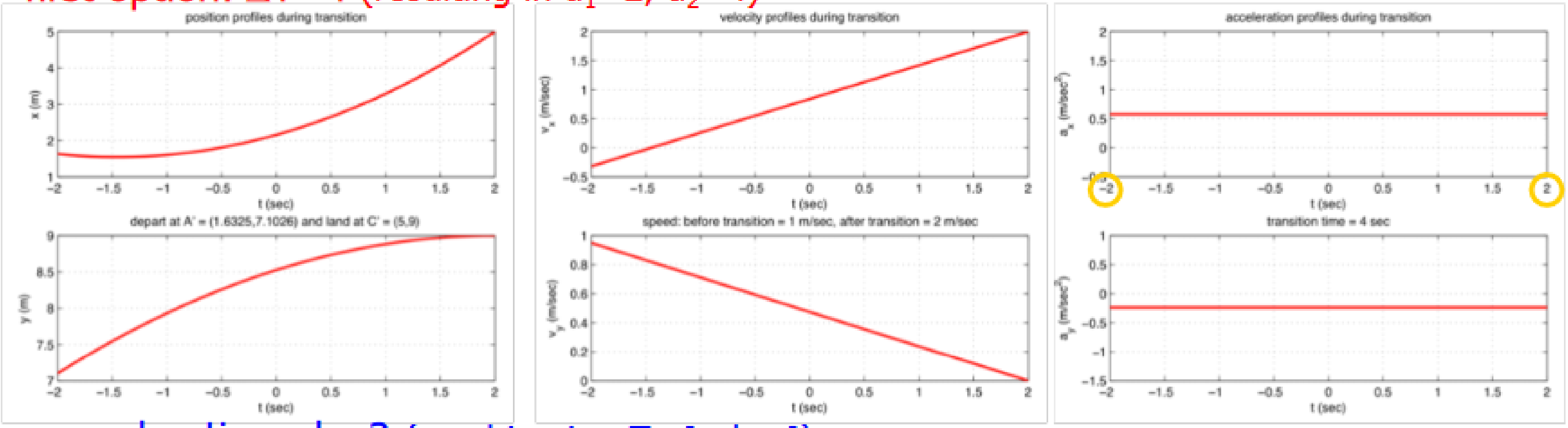
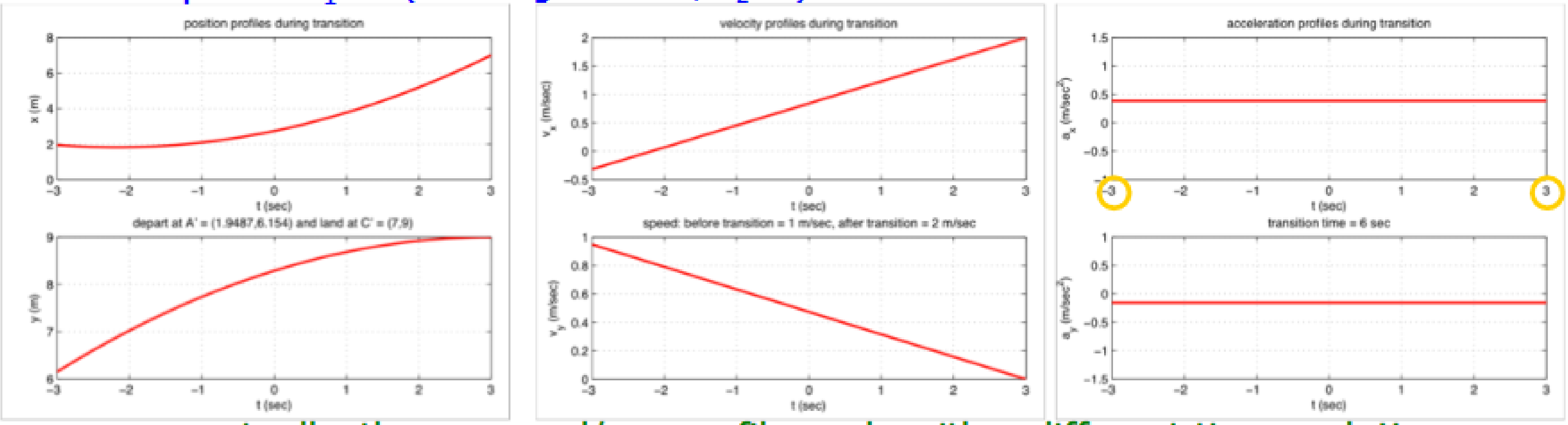
A numerical example

first option: $\Delta T=4$ (resulting in $d_1=2$, $d_2=4$)



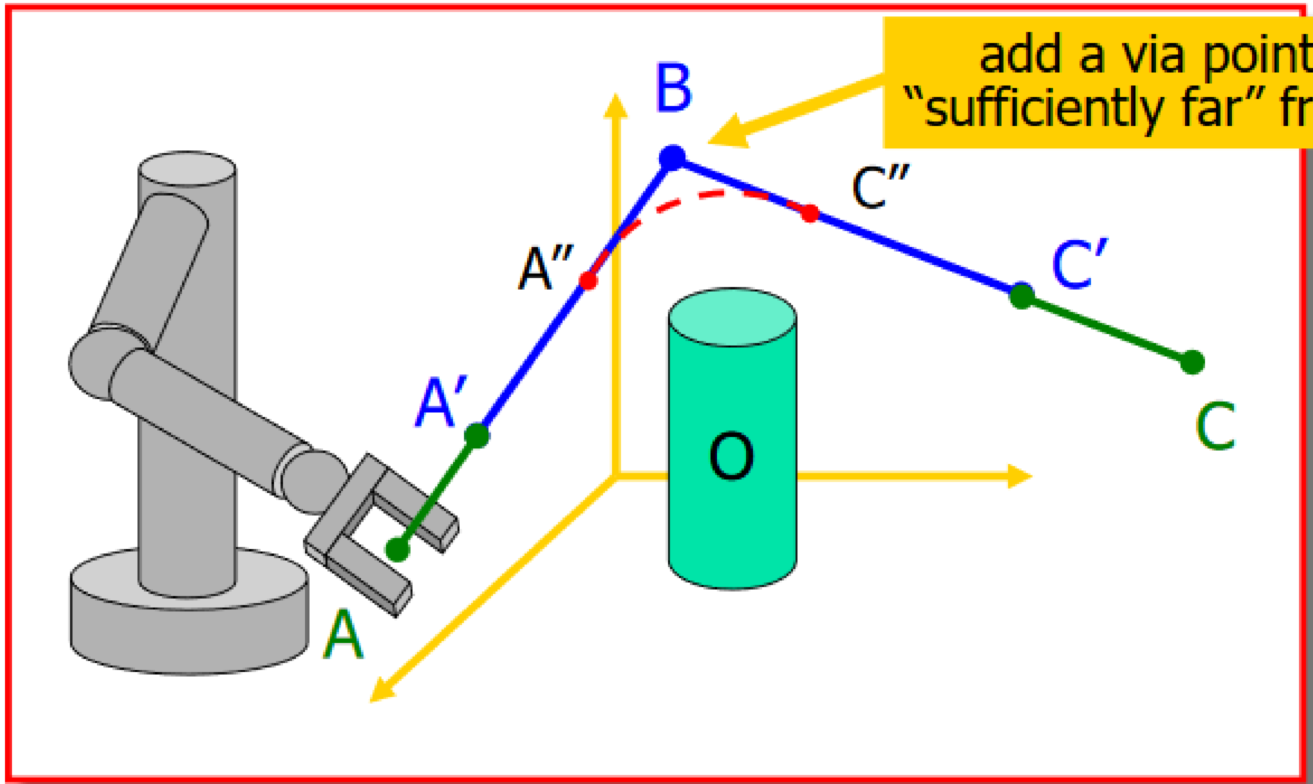
second option: $d_1=3$ (resulting in $\Delta T=6$, $d_2=6$)



actually, the same vel/acc profiles only with a different time scale!!

Application example

plan a Cartesian trajectory from A to C (rest-to-rest)
that avoids the obstacle O, with $a \leq a_{max}$ and $v \leq v_{max}$



add a via point B
"sufficiently far" from O

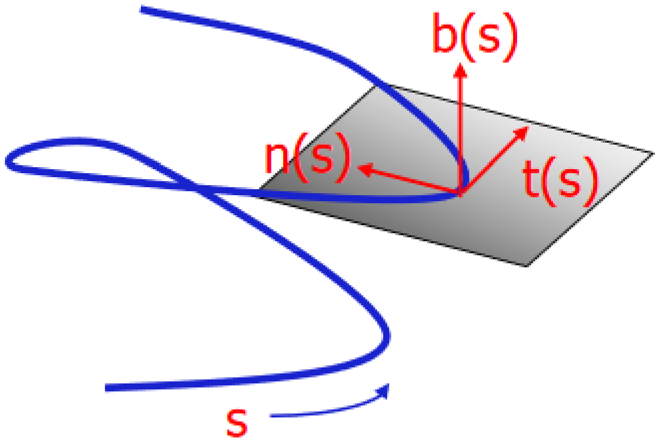
on $\overline{AA'} \rightarrow a_{max}$ on $\overline{A'B}$ and $\overline{BC'} \rightarrow v_{max}$ on $\overline{C'C} \rightarrow -a_{max}$
+ over-fly between A″ e C″

Other Cartesian paths

- circular path through 3 points in 3D (often built-in feature)
- linear path for the end-effector with constant orientation
- in robots with spherical wrist: planning may be decomposed into a path for wrist center and one for E-E orientation, with a common timing law
- though more complex in general, it is often convenient to parameterize the Cartesian geometric path p(s) in terms of its arc length (e.g., with $s = R\theta$ for circular paths), so that
  - velocity: $dp/dt = dp/ds \cdot ds/dt$
    - dp/ds = unit vector ($\|\cdot\| = 1$) tangent to the path: tangent direction t(s)
    - ds/dt = absolute value of tangential velocity (= speed)
  - acceleration: $d^2p/dt^2 = d^2p/ds^2 \cdot (ds/dt)^2 + dp/ds \cdot d^2s/dt^2$
    - $\|d^2p/ds^2\|$ = curvature $\kappa(s)$ (= 1/radius of curvature)
    - $d^2p/ds^2 \cdot (ds/dt)^2$ = centripetal acceleration: normal direction n(s) $\perp$ to the path, on the osculating plane; binormal direction $b(s) = t(s) \times n(s)$
    - d^2s/dt^2 = scalar value (with any sign) of tangential acceleration

Definition of Frenet frame

- For a generic (smooth) path $p(s)$ in R^3, parameterized by s (not necessarily its arc length), one can define a reference frame as in figure



$p' = dp/ds \qquad p'' = d^2p/ds^2$

derivatives w.r.t. the parameter

$t(s) = p'(s)/\|p'(s)\|$
unit tangent vector

$n(s) = p''(s)/\|p''(s)\|$
unit normal vector
($\in$ osculating plane)

$b(s) = t(s) \times n(s)$
unit binormal vector

- general expression of path curvature (at a path point $p(s)$)

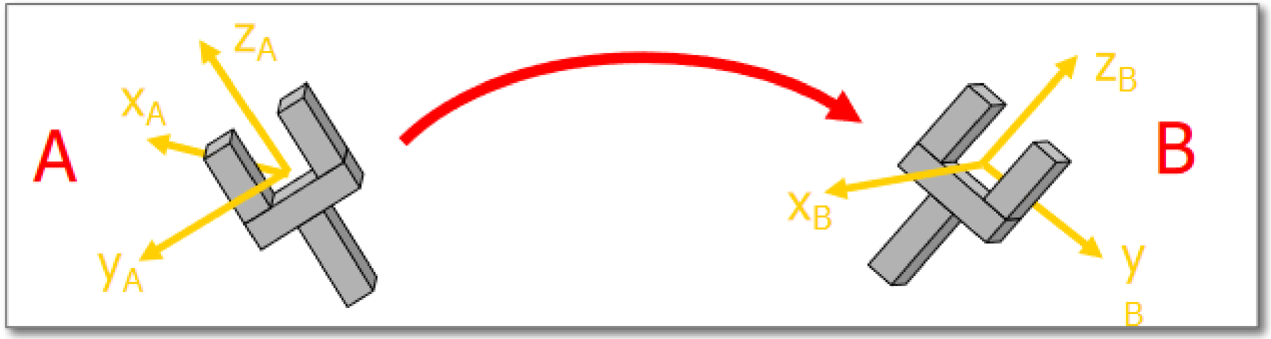
$$\kappa(s) = \|p'(s) \times p''(s)\| / \|p'(s)\|^3$$

Optimal trajectories

- for Cartesian robots (e.g., PPP joints)
  1. the straight line joining two position points in the Cartesian space is one path that can be executed in minimum time under velocity/acceleration constraints (but other such paths may exist, if (joint) motion can also be not coordinated)
  2. the optimal timing law is of the bang-coast-bang type in acceleration (in this special case, also in terms of actuator torques)

- for articulated robots (with at least a R joint)
  - 1. e 2. are no longer true in general in the Cartesian space, but time-optimality still holds in the joint space when assuming bounds on joint velocity/acceleration
    - straight line paths in the joint space do not correspond to straight line paths in the Cartesian space, and vice-versa
  - bounds on joint acceleration are conservative (though kinematically tractable) w.r.t. actual ones on actuator torques, which involve the robot dynamics
    - when changing robot configuration/state, different torque values are needed to impose the same joint accelerations

Planning orientation trajectories



- using minimal representations of orientation (e.g., ZXZ Euler angles ϕ,θ,ψ), we can plan independently a trajectory for each component
  - e.g., a linear path in space $\phi\ \theta\ \psi$, with a cubic timing law

    $\Rightarrow$ but poor prediction/understanding of the resulting intermediate orientations

- alternative method: based on the axis/angle representation
  - determine the (neutral) axis r and the angle θ_{AB}: $R(r,\theta_{AB}) = R_A^T R_B$ (rotation matrix changing the orientation from A to B $\Rightarrow$ inverse axis-angle problem)
  - plan a timing law $\theta(t)$ for the (scalar) angle θ interpolating 0 with θ_{AB} (with possible constraints/boundary conditions on its time derivatives)
  - $\forall t$, $R_A R(r,\theta(t))$ specifies then the actual end-effector orientation at time t

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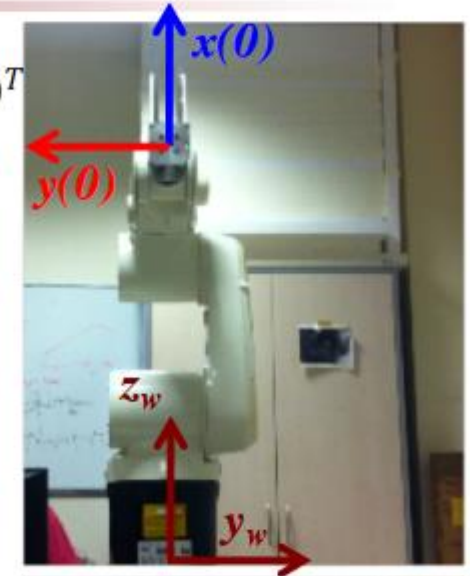
A complete position/orientation Cartesian trajectory

- initial given configuration $q(0) = (0 \quad \pi/2 \quad 0 \quad 0 \quad 0 \quad 0)^T$
- initial end-effector position $p(0) = (0.540 \quad 0 \quad 1.515)^T$
- initial orientation

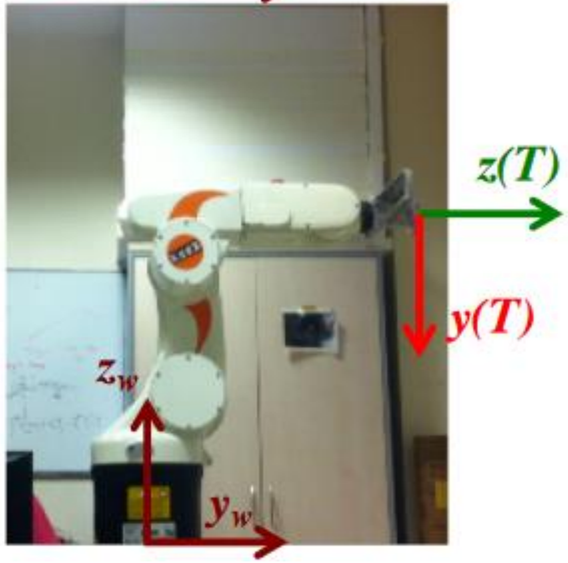
$$R(0) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$



linear path for position ⬇ **axis-angle** method for orientation

- final end-effector position $p(T) = (0 \quad 0.540 \quad 1.515)^T$
- final orientation

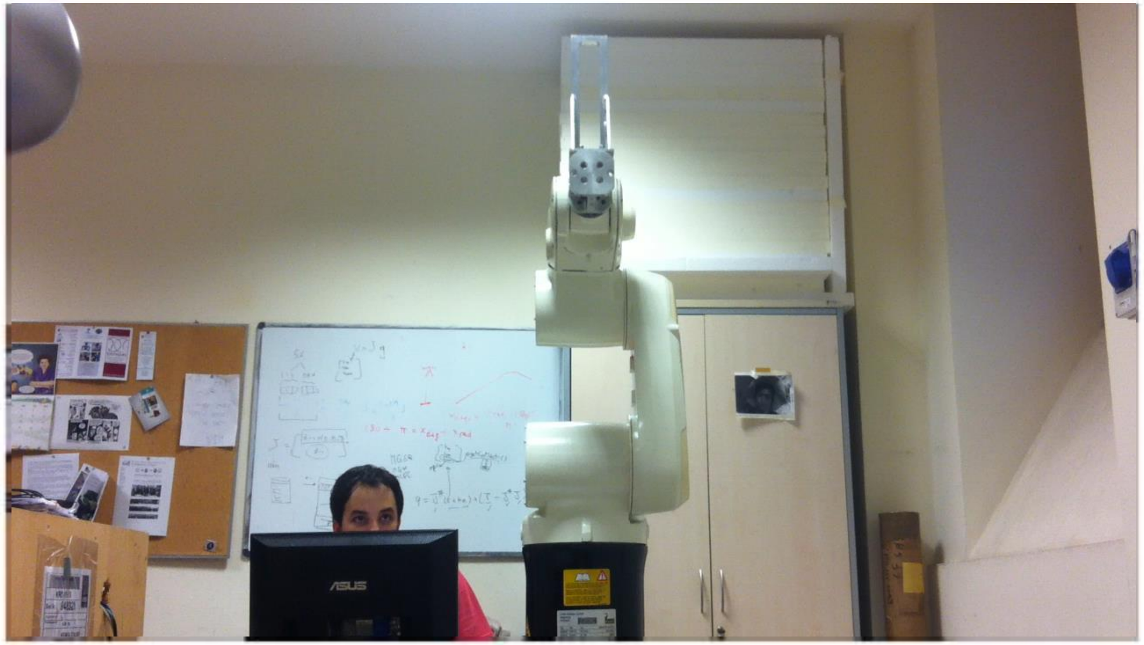
$$R(T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$



- the final configuration is NOT specified a priori

Axis-angle orientation trajectory

video



$$L = \|p_{\text{final}} - p_{\text{init}}\| = 0.763 \text{ [m]}$$

$$\omega = r\dot{\theta} \;\rightarrow\; \|\omega\| = |\dot{\theta}|$$

$$\dot{\omega} = r\ddot{\theta} \;\rightarrow\; \|\dot{\omega}\| = |\ddot{\theta}|$$

coordinated
Cartesian motion
with bounds

$$v_{max} = 0.4 \text{ [m/s]}$$
$$a_{max} = 0.1 \text{ [m/s}^2\text{]}$$
$$\omega_{max} = \pi/4 \text{ [rad/s]}$$
$$\dot{\omega}_{max} = \pi/8 \text{ [rad/s}^2\text{]}$$

triangular speed profile $\dot{s}(t)$ with minimum time $T = 5.52$ s (imposed by the bounds on linear motion)

$$p(s) = p_{\text{init}} + s(p_{\text{final}} - p_{\text{init}}) = (0.540 \quad 0 \quad 1.515)^T + s(-0.540 \quad 0.540 \quad 0)^T, \qquad s \in [0,1]$$

$$\boldsymbol{s = s(t),\ t \in [0, T]}$$

$$R_{\text{init}} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = R_{\text{init}}^T$$

$$R_{\text{final}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$R_{\text{init}}^T R_{\text{final}} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix} = Rot(r, \theta_{if})$$

$$r = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \theta_{if} = \frac{2\pi}{3} \text{ [rad]} (= 120°)$$

$$R(s) = R_{\text{init}} Rot(r, \theta(s))$$

$$\theta(s) = s\theta_{if}, \qquad s \in [0,1]$$

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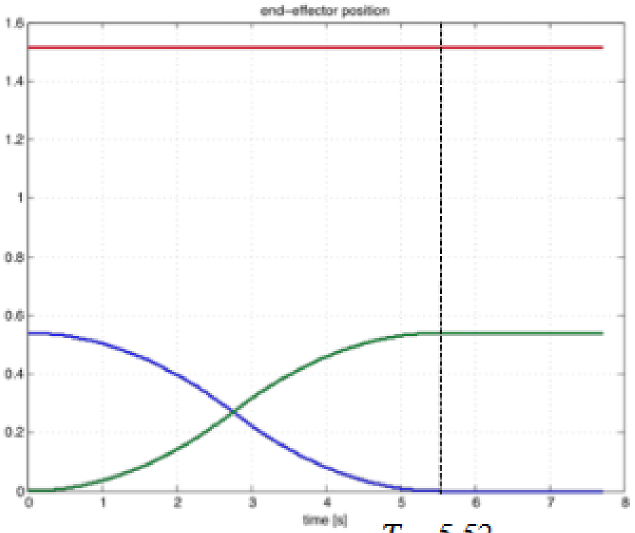
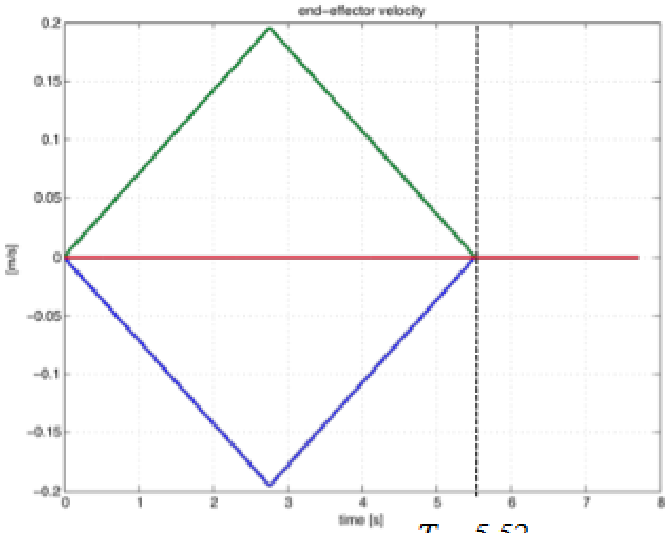
Axis-angle orientation trajectory

x = —
y = —
z = —

planned motion of Cartesian position and velocity



$T = 5.52$

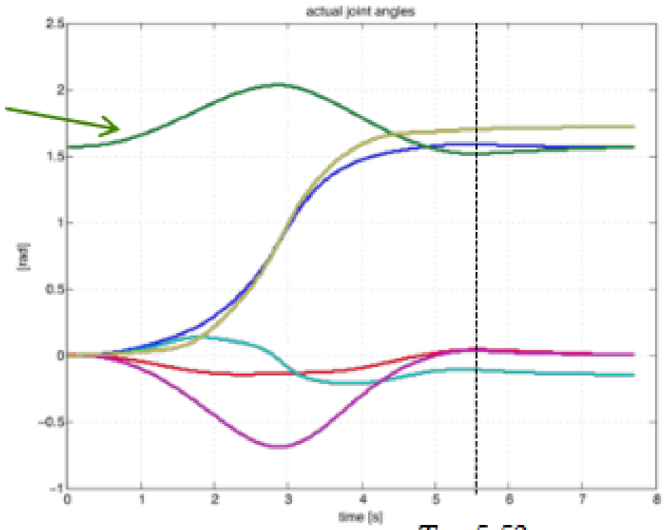


$T = 5.52$

triangular profile for linear speed $T = 5.52$ s

joint 2

actual joint motion



$T = 5.52$

- the robot joint velocity was commanded by inversion of the **geometric** Jacobian
- a **user** program, via KUKA RSI interface at T_c = **12** ms sampling time (one-way communication)
- robot motion execution is ≈ what was planned, but only thanks to an external **kinematic control** loop (at **task** level)

Comparison of orientation trajectories
Euler angles vs. axis-angle method

- initial configuration $q(0) = (0 \quad \pi/2 \quad \pi/2 \quad 0 \quad -\pi/2 \quad 0)^T$
- initial end-effector position $p(0) = (0.115 \quad 0 \quad 1.720)^T$
- initial orientation

$$R(0) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

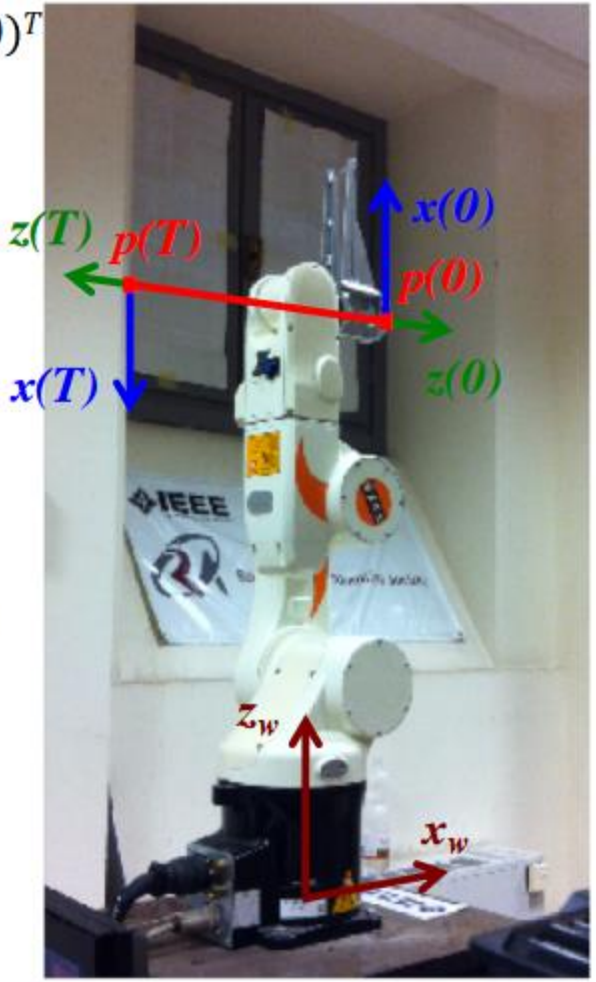
- initial Euler ZYZ angles $\phi_{ZYZ}(0) = (0 \quad \pi/2 \quad \pi)^T$

via a **linear path** (for position)

- final end-effector position $p(T) = (-0.172 \quad 0 \quad 1.720)^T$
- final orientation

$$R(T) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

- final Euler ZYZ angles $\phi_{ZYZ}(T) = (-\pi \quad \pi/2 \quad 0)^T$



Comparison of orientation trajectories

Euler angles vs. axis-angle method

$$R_{\text{init}} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \phi_{ZYZ,\text{init}} = \begin{pmatrix} 0 \\ \pi/2 \\ \pi \end{pmatrix}$$

$$R_{\text{final}} = -\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \phi_{ZYZ,\text{final}} = \begin{pmatrix} -\pi \\ \pi/2 \\ 0 \end{pmatrix}$$

video

using ZYZ Euler angles

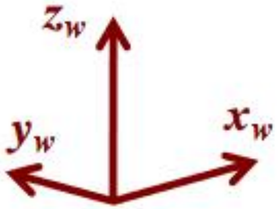
using axis-angle method

video

$$R_{\text{init}}^T R_{\text{final}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\Rightarrow r = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix},$$

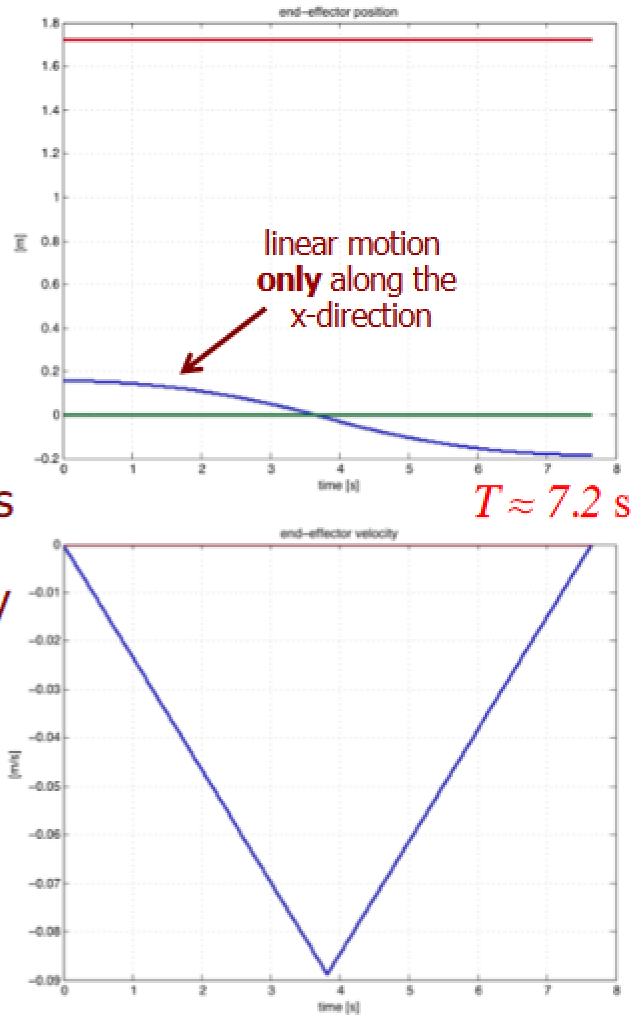
$$\theta = \pi$$



Comparison of orientation trajectories
Euler angles vs. axis-angle method

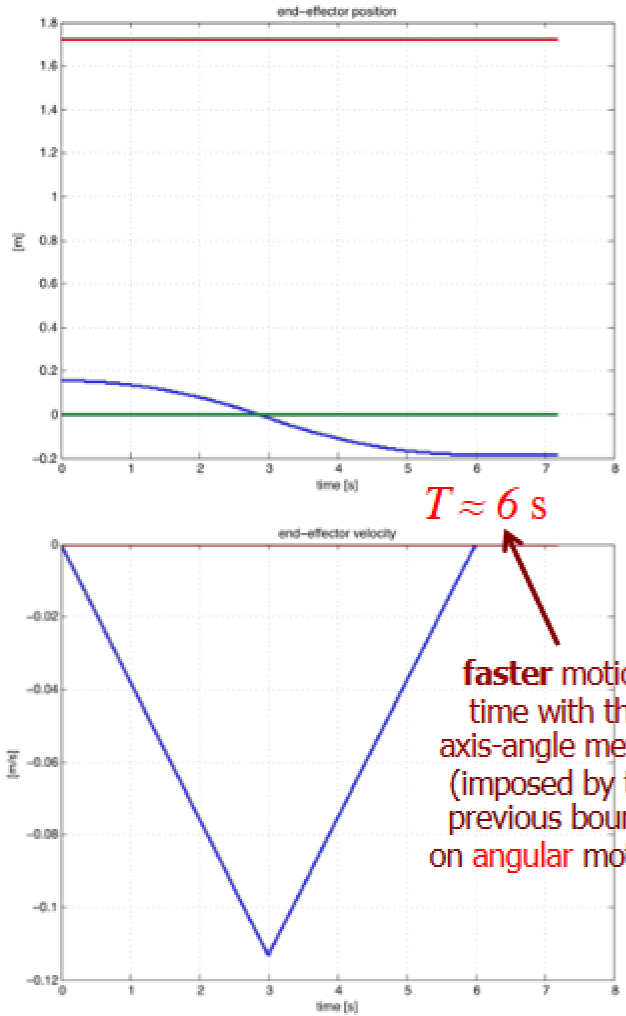
x = —— (blue)
y = —— (green)
z = —— (red)

planned Cartesian components of position and velocity



$T \approx 7.2$ s

using ZYZ Euler angles



$T \approx 6$ s

faster motion time with the axis-angle method (imposed by the previous bounds on angular motion)

using axis-angle method

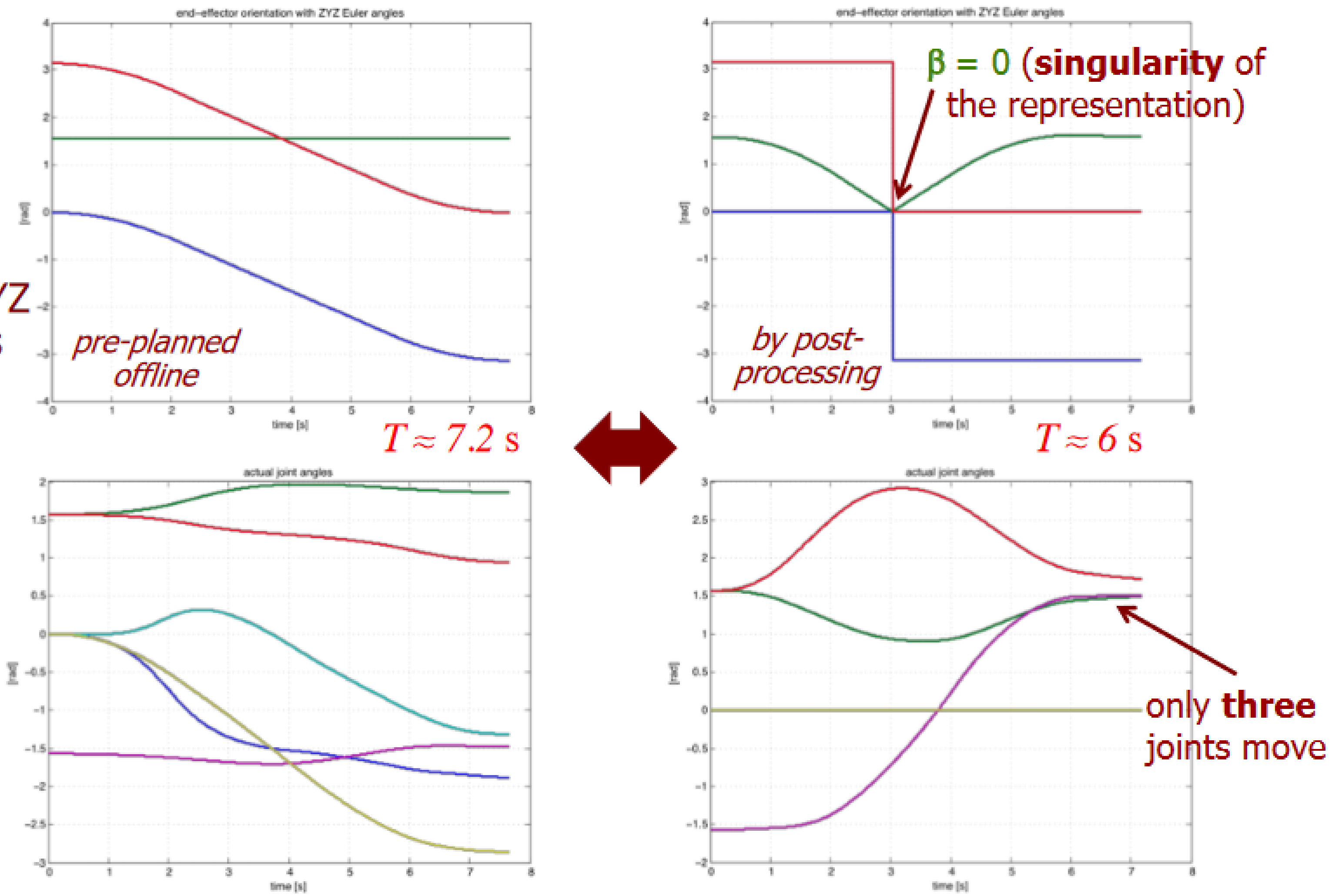
Comparison of orientation trajectories

Euler angles vs. axis-angle method

α = —
β = —
γ = —

orientation in terms of ZYZ Euler angles

actual joint motion



using ZYZ Euler angles

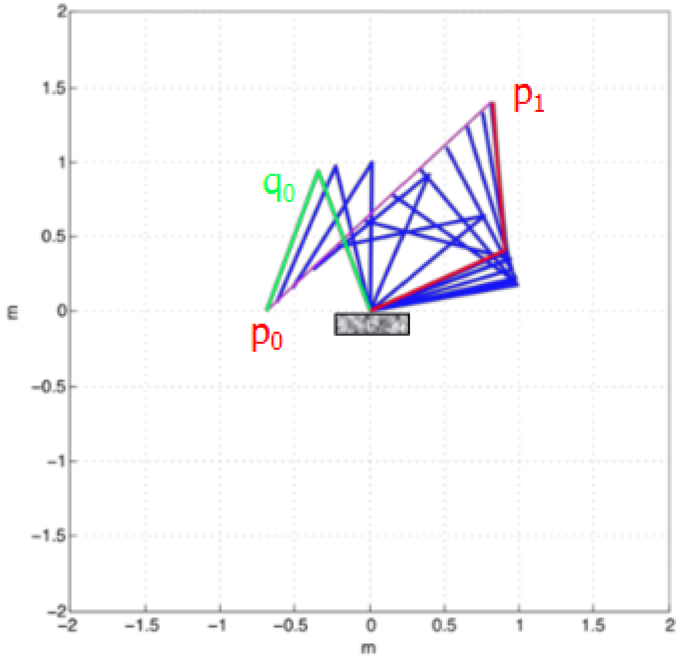
using axis-angle method

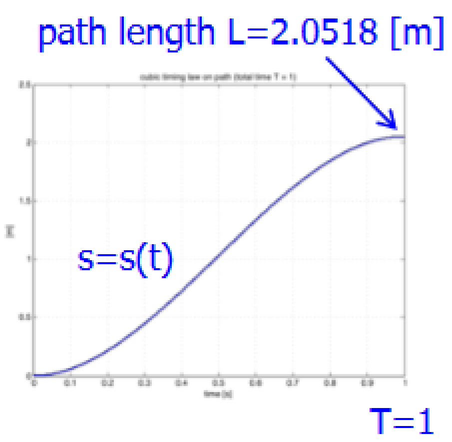
Uniform time scaling

- for a given path p(s) (in joint or Cartesian space) and a given timing law $s(\tau)$ ($\tau=t/T$, T="motion time"), we need to check if existing bounds v_{max} on (joint) velocity and/or a_{max} on (joint) acceleration are violated or not
  - … unless such constraints have already been taken into account during the trajectory planning, e.g., by using a bang-coast-bang acceleration timing law
- velocity scales **linearly** with motion time
  - $dp/dt = dp/ds \cdot ds/d\tau \cdot 1/T$
- acceleration scales **quadratically** with motion time
  - $d^2p/dt^2 = (d^2p/ds^2 \cdot (ds/d\tau)^2 + dp/ds \cdot d^2s/d\tau^2) \cdot 1/T^2$
- if motion is unfeasible, scale (increase) time $T \rightarrow kT$ ($k>1$), based on the "most violated" constraint (max of the ratios $|v|/v_{max}$ and $|a|/a_{max}$)
- if motion is "too slow" w.r.t. the robot capabilities, decrease T ($k<1$)
  - in both cases, after scaling, there will be (at least) one instant of saturation (for at least one variable)
  - no need to re-compute motion profiles from scratch!

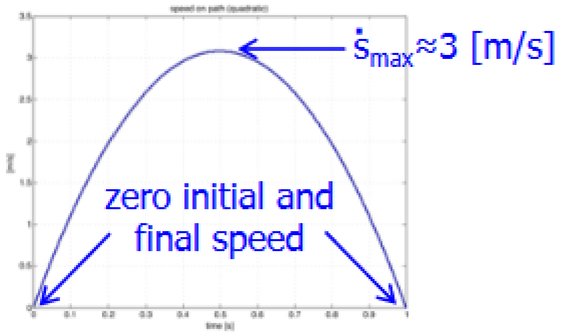
Numerical example - 1

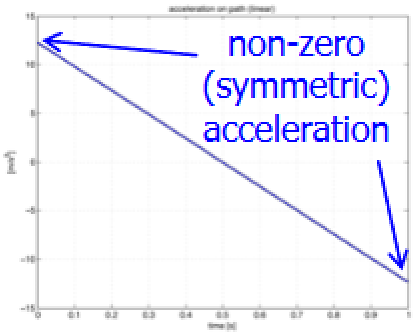
- 2R planar robot with links of unitary length (1 [m])
- linear Cartesian path p(s) from $q_0=(110°, 140°) \Rightarrow p_0=f(q_0)=(-.684, 0)$ [m] to $p_1=(0.816, 1.4)$, with rest-to-rest cubic timing law s(t), T=1 [s]
- bounds in joint space: max (absolute) velocity $v_{max,1}= 2$, $v_{max,2}= 2.5$ [rad/s], max (absolute) acceleration $a_{max,1}= 5$, $a_{max,2}= 7$ [rad/s^2]





Cubic spline

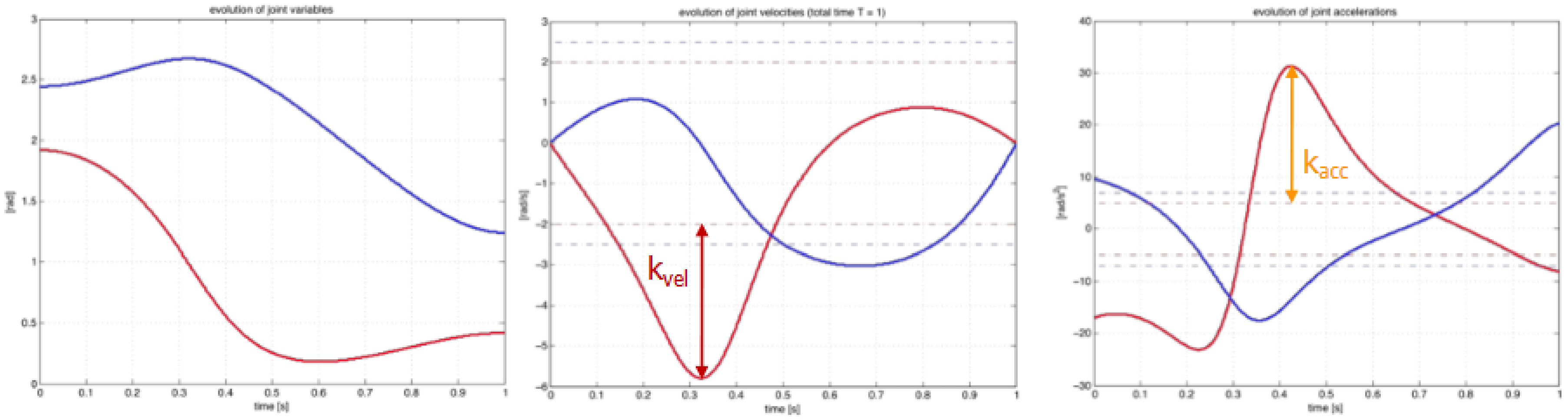




Numerical example - 2

- violation of both joint velocity and acceleration bounds with T=1 [s]
  - max relative violation of joint velocities: $k_{vel} = 2.898 = \max\{1, |\dot{q}_1|/v_{max,1}, |\dot{q}_2|/v_{max,2}\}$
  - max relative violation of joint accelerations: $k_{acc} = 6.2567 = \max\{1, |\ddot{q}_1|/a_{max,1}, |\ddot{q}_2|/a_{max,2}\}$
- minimum uniform time scaling of Cartesian trajectory to recover feasibility

$$k = \max\{1, k_{vel}, \sqrt{k_{acc}}\} = 2.898 \quad \Rightarrow \quad T_{scaled} = kT = 2.898 > T$$



—— = joint 1 —— = joint 2

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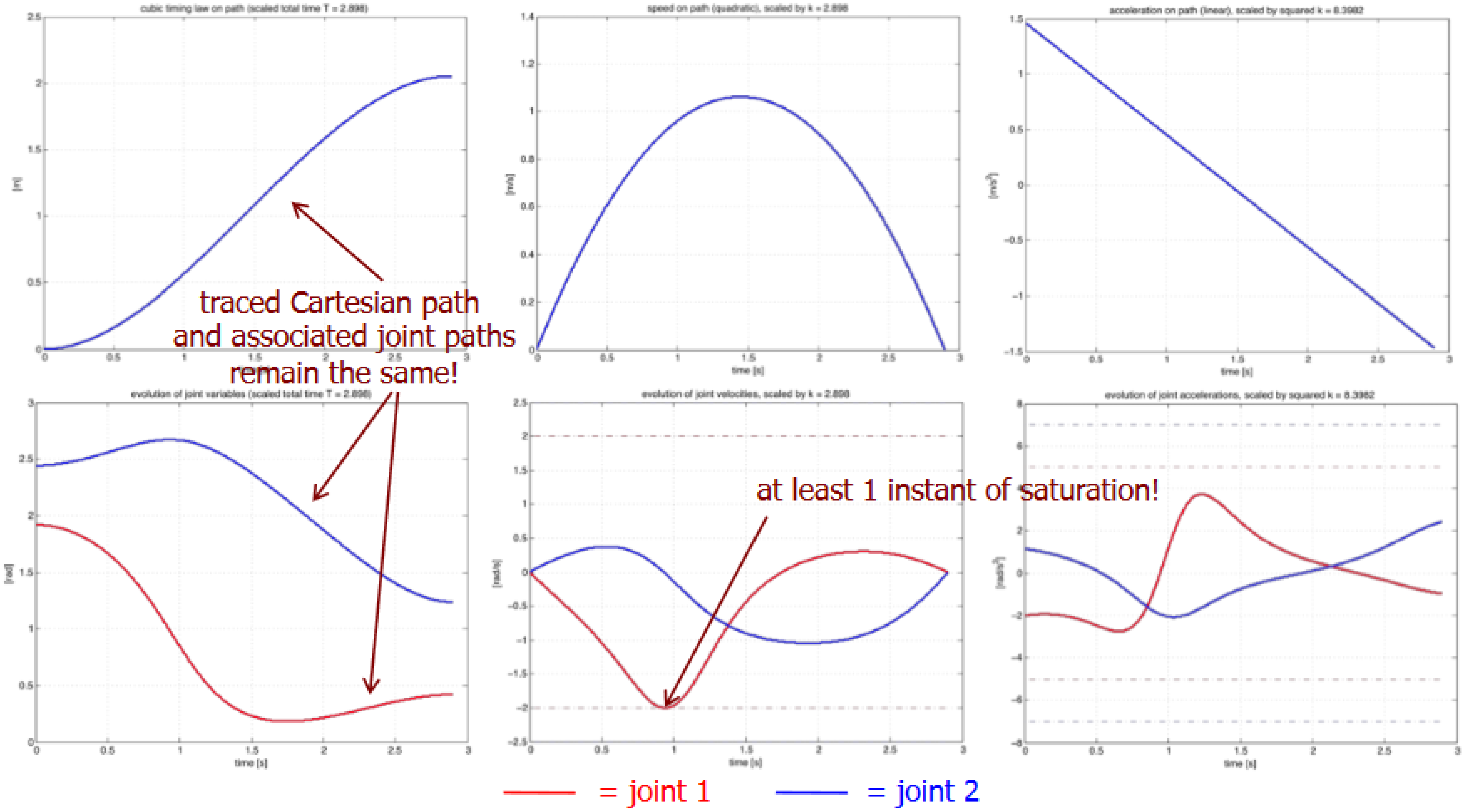
Numerical example - 3

- scaled trajectory with $T_{scaled} = 2.898$ [s]
  - speed [acceleration] on path and joint velocities [accelerations] scale linearly [quadratically]



The end!